Relativistic Mechanics

Introduction

We saw that the Galilean transformation in classical mechanics is incorrect in view of special relativity. The Lorentz transformation correctly describe how the space and time coordinates of an event transform for inertial frames of reference. We also saw that the simple velocity addition law that results from the Galilean transformation is incorrect, and velocities are seen to transform according to rules consistent with the Lorentz transformation. These results should make one think about the structure of classical mechanics itself, and whether all the laws of classical mechanics, including such seemingly immutable ones as conservation of momentum and conservation of energy, need to be reformulated. We now take a look at such quantities as mass, velocity, momentum, and energy in the context of special relativity, and build a new, relativistic mechanics based upon this framework.

Relativistic Momentum and Energy

First, we take a look at momentum and its conservation. Note that momentum itself is not of any special importance in classical mechanics. However, momentum conservation is a very important principle which one would certainly like to retain in special relativity. We now show that the law of conservation of momentum is inconsistent with special relativity if the classical definition of momentum is maintained.¹



Classically, momentum is conserved in the collision process depicted above:

$$n_1 u_1 + m_2 u_2 = m_3 u_3 + m_4 u_4. aga{3.1}$$

We are assuming a one-dimensional collision to simplify the discussion. (Thus, we have dropped the subscript "x".) Also, we have assumed that the masses can change as a result of the collision (interaction). Now according to the classical velocity transformation, u = u' + v, where u' is the velocity in an inertial frame S' moving relative to S with velocity v. Using this transformation, one finds that

$$m_1 u_1' + m_2 u_2' = m_3 u_3' + m_4 u_4', \tag{3.2}$$

if

$$m_1 + m_2 = m_3 + m_4. \tag{3.3}$$

Thus momentum is also conserved in S' if mass is simultaneously conserved.

However, if one uses the (correct) relativistic transformation, one finds that momentum is not conserved. We now show this explicitly. If we assume momentum is conserved in the S frame, then using

¹ See Griffiths' Introduction to Electrodynamics, 3rd Ed., Ch. 12

$$u_x = \frac{u'_x + v}{1 + \frac{v u'_x}{c^2}},$$
(3.4)

we find that

$$m_1\left(\frac{u_1'+v}{1+\frac{vu_1'}{c^2}}\right) + m_2\left(\frac{u_2'+v}{1+\frac{vu_2'}{c^2}}\right) = m_3\left(\frac{u_3'+v}{1+\frac{vu_3'}{c^2}}\right) + m_4\left(\frac{u_4'+v}{1+\frac{vu_4'}{c^2}}\right).$$
(3.5)

It is quite clear from Eq. (3.5) that classical momentum is not conserved in the S' frame. In order to preserve momentum conservation, we must redefine momentum.

Classically, $\vec{p}_{cl} = m\vec{u} = m\frac{d\vec{r}}{dt}$. [Note that the subscript "cl" denotes "classical."] All quantities are measured in a single inertial frame, say S. Let us define the proper velocity \vec{v} by

$$\vec{v} = \frac{d\vec{r}}{dt_0},\tag{3.6}$$

where dt_0 is the proper time, i.e., the time interval as measured in the rest frame of the moving particle. We define the relativistic momentum \vec{p} by

$$\vec{p} = m\vec{v}, \qquad (3.7)$$



where the mass *m* is the mass measured in the *rest frame* of the moving particle. The mass *m* is called the *rest mass* of the particle. Now, we know that $dt = \gamma dt_0$ and therefore $\vec{v} = \gamma \vec{u}$. Hence

$$\vec{p} = \gamma m \vec{u}, \tag{3.8}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}.$$
(3.9)

Note that the classical momentum is obtained from the relativistic momentum when $\gamma \approx 1$. Squaring Eq. (3.8) (i.e., taking the dot product of \vec{p} with itself), we obtain

$$p^{2} = \frac{m^{2}u^{2}}{1 - \frac{u^{2}}{c^{2}}},$$
(3.10)

which leads to

$$u = \frac{c}{\sqrt{1 + \left(\frac{mc}{p}\right)^2}}.$$
(3.11)

We see that despite the fact that p can increase without limit (due to γ), u is always less than c.

How does the proper velocity transform? It turns out that the proper velocity transforms much more simply than the "ordinary" velocity. The reason for the complicated expression for the relativistic velocity transformation is that both numerator and denominator of the expression $\frac{dx}{dt}$ have to be transformed. In the case of the proper velocity, only the numerator needs to be transformed, since the proper time is *invariant*. Let us assume one-dimensional motion along the *x* axis. We have

$$v_x = \frac{dx}{dt_0} = \frac{\gamma_v \left(dx' + v dt' \right)}{dt_0},$$
(3.12)

where

$$\gamma_{\nu} = \frac{1}{\sqrt{1 - \frac{\nu^2}{c^2}}},$$
(3.13)

and v is the relative speed of the inertial frames S and S'. Eq. (3.12) gives

$$\boldsymbol{v}_{x} = \boldsymbol{\gamma}_{v} \left(\boldsymbol{v}_{x}^{\prime} + v \boldsymbol{\gamma} \right), \tag{3.14}$$

where,

$$\gamma = \frac{dt'}{dt_0} = \frac{1}{\sqrt{1 - \frac{{u'}^2}{c^2}}}.$$
(3.15)

Now, let us assume conservation of momentum holds in the S frame:

$$m_1 v_1 + m_2 v_2 = m_3 v_3 + m_4 v_4. aga{3.16}$$

(One-dimensional motion, so subscript "x" has been dropped.) Using the transformation of the proper velocity [Eq. (3.14)], Eq. (3.16) becomes

$$m_1 \left[\gamma_{\nu} \left(\nu_1' + \nu \gamma_1 \right) \right] + m_2 \left[\gamma_{\nu} \left(\nu_2' + \nu \gamma_2 \right) \right] = m_3 \left[\gamma_{\nu} \left(\nu_3' + \nu \gamma_3 \right) \right] + m_4 \left[\gamma_{\nu} \left(\nu_4' + \nu \gamma_4 \right) \right].$$
(3.17)

Canceling common factors, we see that conservation of momentum holds in the S' frame:

$$m_1 v_1' + m_2 v_2' = m_3 v_3' + m_4 v_4', (3.18)$$

if

$$\gamma_1 m_1 + \gamma_2 m_2 = \gamma_3 m_3 + \gamma_3 m_3 \tag{3.19}$$

is simultaneously true. Hence, (rest) mass is no longer conserved; instead, the quantity γm is conserved. The real significance of γm is its relationship to the energy of the particle. The quantity

$$E = \gamma mc^2 \tag{3.20}$$

is defined to be the *relativistic energy* of the particle. Obviously, Eq. (3.19) implies that relativistic energy is conserved. We see the overall picture now. If momentum is defined as $\vec{p} = \gamma m \vec{u}$, then momentum conservation is consistent with special relativity, provided that the relativistic energy $E = \gamma m c^2$ is also conserved. (This is also true in two or three spatial dimensions, though we have not explicitly proved it.)

In every closed (no external forces) system, the total relativistic energy and momentum are conserved. This law has been experimentally verified in numerous cases.

Why do we call the quantity E the (total) energy? Well, it has the correct dimensions, for one thing (number \times mass \times speed squared, like kinetic energy). More compellingly, let us see how E behaves for speeds that are small compared to c.

$$E = \gamma mc^{2} = \frac{1}{\sqrt{1 - \frac{u^{2}}{c^{2}}}} mc^{2}.$$
 (3.21)

Now,

$$\left(1 - \frac{u^2}{c^2}\right)^{-1/2} \approx 1 + \frac{1}{2} \frac{u^2}{c^2}.$$
(3.22)

Thus,

$$E \approx mc^2 + \frac{1}{2}mu^2. \tag{3.23}$$

The second term is just the classical kinetic energy. The first term is called the *rest energy*. It is the total energy of an object that is at rest relative to an inertial frame.

We can define the relativistic kinetic energy *K* in the following way:

$$E = mc^2 + K. \tag{3.24}$$

Since $E = \gamma mc^2$, then

$$K = (\gamma - 1)mc^2$$
. (3.25)

Example: Calculate the rest energy of an electron. Solution: $E_0 = mc^2 = (9.109 \times 10^{-31} \text{ kg}) (2.998 \times 10^8 \text{ m/s})^2 = 8.187 \times 10^{-14} \text{ J}.$

A more convenient unit is the electron volt: $1 \text{ eV} = 1.60 \times 10^{-19} \text{ J}$. Thus, for the electron,

$$E_0 = \frac{8.187 \times 10^{-14} \text{ J}}{1.60 \times 10^{-19} \text{ J/eV}} = 5.11 \times 10^5 \text{ eV} = 0.511 \text{ MeV}.$$

Relations Between Energy and Momentum

Let us now derive the relationship between *E* and *p*. Now, $\vec{p} = \gamma m \vec{u}$ and $E = \gamma m c^2$. Thus,

$$\frac{\vec{p}}{E} = \frac{\vec{u}}{c^2},\tag{3.26}$$

i.e.,

$$\frac{\vec{p}c}{E} = \frac{\vec{u}}{c}.$$
(3.27)

Considering the magnitudes of the vector quantities gives the useful relation

$$\beta = \frac{u}{c} = \frac{pc}{E}.$$
(3.28)

If we square Eq. (3.20), we find

$$E^{2} = \gamma^{2} m^{2} c^{2} = \frac{m^{2} c^{2}}{1 - \frac{u^{2}}{c^{2}}}.$$
(3.29)

Squaring Eq. (3.28) and substituting in Eq. (3.29) gives another useful relation: $E^{2} = p^{2}c^{2} + m^{2}c^{4}.$ (3.30) This relation allows one to determine E if p is known or vice-versa, without having to calculate the velocity.

Example: An electron is accelerated through a potential difference of 10^6 V. Find the electron's momentum.

Solution:

The kinetic energy gained by the electron is given by $K = (e)(V) = (e)(10^6 \text{ Volts}) = 1.00 \text{ MeV}.$ Now, $E = E_0 + K$, where E_0 is the rest energy of the electron. Squaring, we find $E^2 = E_0^2 + 2E_0K + K^2$. But, $E^2 = p^2 c^2 + E_0^2$. Hence, upon equating the two expressions, we find $p^2 c^2 = 2E_0K + K^2$,

or,

$$pc = \sqrt{2E_0K + K^2}.$$

Now, K = 1.00 MeV and $E_0 = 0.511$ MeV. Substituting yields pc = 1.42 MeV, i.e., p = 1.42 MeV/c.

Problem 31, Chapter, TZD, 2/e.





Conservation of Energy: $E = E_1 + E_2$

$$= \sqrt{(p_1c)^2 + (m_1c^2)^2} + \sqrt{(p_2c)^2 + (m_2c^2)^2}$$

= $\sqrt{(2 \text{ GeV})^2 + (0.5 \text{ GeV})^2} + \sqrt{(1.5 \text{ GeV})^2 + (1 \text{ GeV})^2}$
= 3.86 GeV.

Conservation of momentum:

 $\vec{p} = \vec{p}_1 + \vec{p}_2.$ Because \vec{p}_1 and \vec{p}_2 are perpendicular, $p^2 = p_1^2 + p_2^2,$ i.e., $p = \sqrt{p_1^2 + p_2^2} = \sqrt{(2 \text{ GeV}/c)^2 + (1.5 \text{ GeV}/c)^2}$ = 2.5 GeV/c.

Now, for the original particle, $E^2 = p^2 c^2 + m^2 c^4$, or. $mc^{2} = \sqrt{E^{2} - p^{2}c^{2}} = \sqrt{(3.86 \text{ GeV})^{2} - (2.5 \text{ GeV})^{2}}$ = 2.94 GeV.Thus, $m = 2.95 \text{ GeV}/c^2$. $\beta = \frac{pc}{E} = \frac{2.5 \text{ GeV}}{3.86 \text{ GeV}} = 0.648.$

Hence, u = 0.648c.

Mass-Energy Equivalence

From the relation $E_0 = mc^2$, we see that the rest mass *m* of an object is *equivalent* to an amount of energy $E_0 = mc^2$. Conversely, an energy change ΔE that occurs in an interaction produces a rest mass change $\Delta m = \Delta E / c^2$. It follows that in interactions in which kinetic energy is not conserved (inelastic processes), the rest mass of the system will change. Consider a simple example: a bomb of rest mass M explodes into two pieces, each having the same rest mass and traveling in opposite directions with speed (4/5)c.



Relativistic momentum is obviously conserved. Total energy E_i before the explosion = Mc^2 . Total energy E_f after the explosion is given by $E_f = 2\gamma mc^2$ (since the particles have the same masses and speeds). Thus,

$$E_f = \frac{2mc^2}{\sqrt{1 - \left(\frac{4}{5}\right)^2}} = \frac{10}{3}mc^2.$$

By conservation of energy-momentum, $E_i = E_f$, i.e., $Mc^2 = \frac{10}{2}mc^2$,

$$Mc^2 = \frac{10}{3}mc^2$$

i.e.,

$$M = \frac{10}{3}m.$$

Note that M > 2m, i.e., the total rest mass decreased. A part of the initial rest mass of the bomb was converted to kinetic energy of the fragments. This extra initial rest mass was due to the chemical energy of the bomb. The internal energy of a body is the kinetic energy of random motion, plus the potential energy of all the atoms. From the point of view of relativity, the internal energy adds to the rest mass of the body. Therefore, a hotter body has a greater total rest mass than when it is cooler; a stretched spring has more rest mass than when it is relaxed, etc. Also, emission of radiation decreases rest mass. Such differences in rest mass are extremely small and practically undetectable. However, in nuclear processes, changes in the rest mass are readily measured.

Example:

A deuteron is a bound state consisting of a proton and a neutron. The rest mass of the deuteron is 2.01360 atomic mass units (1 u = 1.66×10^{-27} kg = 931.5 MeV/ c^2). The rest mass of a proton is 1.00728 u and that of a neutron is 1.00867 u. Find the binding energy of the deuteron. [Do alpha particle (He-4 nucleus) in class. Rest mass = 4.001506 u.]

Solution:

Note that since the deuteron is a bound state, energy is required to separate its constituents. The energy needed to disassociate the deuteron increases the rest mass of the constituents. Hence, one expects the rest mass of the deuteron to be less than the total rest mass of its constituents. The difference in rest mass (times c^2) represents the binding energy. From another perspective, when the deuteron is formed, energy is released since the bound state is a lower energy state. This released energy results in a decrease in the rest mass of the deuteron. Quantitatively:

$$\Delta m = \left[(1.00728 + 1.00867) - 2.01360 \right] u = 0.00235 u = 0.00235 \times 931.5 \text{ MeV}/c^2$$

Thus, the binding energy is $\Delta E = (\Delta m)c^2 = 0.00235 \times 931.5 \text{ MeV} = 2.19 \text{ MeV}.$

Massless Particles

In classical physics, particles that have no mass have no momentum and no kinetic energy. Thus, they are dynamically rather uninteresting. Relativity, however, permits zero-rest mass particles with non-zero momentum and energy. Let us see how.

Recall that
$$E^2 = p^2 c^2 + m^2 c^4$$
. If $m = 0$, then
 $E = pc$. [Massless particle.] (3.31)

Further, we know that $\beta = \frac{u}{c} = \frac{pc}{E}$. Since E = pc, it follows that $\beta = 1$, i.e., u = c. [Massless particle]

(3.32)

Thus, a massless particle must travel at the speed of light. An example of a massless particle is the *photon*, which we shall study in detail soon.

What is the momentum of such a particle? Can we calculate it? Let us use the equation $p = \gamma m u$. Substituting for γ gives

$$p = \frac{mu}{\sqrt{1 - \frac{u^2}{c^2}}}.$$

Since u = c and m = 0, we find that p = 0/0, which is indeterminate. Thus, relativity is not inconsistent with a massless particle having a non-zero momentum and energy. Experimentally, the photon has been shown to have non-zero momentum and energy.

[Relativity seems to be asymmetric regarding mass-energy: One can have energy without mass, but there can never be mass without energy.]

Relativistic Dynamics

It is easily seen from the Lorentz transformation that Newton's first law is valid: since inertial frames move at constant velocity relative to one another, if \vec{u} is a constant, so is $\vec{u'}$. (Actually, the first postulate demands that Newton's first law is valid.) What about the second law? The second law written as

$$\vec{F} = \frac{d\vec{p}}{dt} \tag{3.33}$$

is also valid relativistically, provided that \vec{p} represents the relativistic momentum. We can show this by retrieving the work-energy theorem in the context of relativity.

Consider a net force exerted on a single particle. The work done is calculated as

$$W = \int_{initial}^{jinal} \vec{F} \cdot d\vec{s} = \int \frac{d\vec{p}}{dt} \cdot d\vec{s} = \int \frac{d\vec{p}}{dt} \cdot \frac{d\vec{s}}{dt} dt = \int \frac{d\vec{p}}{dt} \cdot \vec{u} dt.$$
(3.34)

Let us show that $\frac{dE}{dt} = \frac{d\vec{p}}{dt} \cdot \vec{u}$.

Now, $E^{2} = p^{2}c^{2} + m^{2}c^{4}.$ Taking the time derivative gives $2E\frac{dE}{dt} = \frac{d}{dt} \Big[(\vec{p} \cdot \vec{p})c^{2} + m^{2}c^{4} \Big] = 2c^{2}\vec{p} \cdot \frac{d\vec{p}}{dt}.$ Thus, $\frac{dE}{dt} = \frac{c^{2}}{E}\vec{p} \cdot \frac{d\vec{p}}{dt} = \vec{u} \cdot \frac{d\vec{p}}{dt}, \text{ using Eq. (3.27). Thus, Eq. (3.34) becomes}$ $W = \int_{initial}^{final} \frac{dE}{dt} dt = \int_{initial}^{final} dE = \int_{E_{i}}^{E_{f}} dE = E_{f} - E_{i}.$ Hence $W = E_{f} - E_{i}.$ (Work-energy theorem). (3.35)

Since $E = mc^2 + K$, Eq. (3.35) can also be written as

$$W = K_f - K_i$$
. (Work-energy theorem). (3.36)

Example

Particle of mass *m* is subjected to a constant force *F*. The particle starts from rest as t = 0. Find position *x* as a function of time. (Taken from Griffiths' *Electrodynamics*, 2nd Ed.)

$$\frac{dp}{dt} = F. \text{ Integrating over time yields } \int \frac{dp}{dt} dt = \int F dt,$$

i.e., $\int_{0}^{p} dp = F \int_{0}^{t} dt'$, which gives $p = Ft$.
Now, $p = \frac{mu}{\sqrt{1 - \frac{u^{2}}{c^{2}}}}$. Thus, $\frac{mu}{\sqrt{1 - \frac{u^{2}}{c^{2}}}} = Ft$. Solving for u gives
 $u = \frac{(F/m)t}{\sqrt{1 + (Ft/mc)^{2}}}.$ ----- (i)

Note that for $t \leq mc/F$, $u \approx (F/m)t$, which is the classical result. Also, as $t \to \infty$, $u \to c$.

To obtain the position x as a function of time, we need to integrate the velocity as given by Eq. (i) over time. Thus

$$x = \frac{F}{m} \int_{0}^{t} \frac{t'dt'}{\sqrt{1 + (Ft'/mc)^{2}}} = \left(\frac{mc^{2}}{F}\right) \sqrt{1 + (Ft'/mc)^{2}} \Big|_{0}^{t}$$
$$= \left(\frac{mc^{2}}{F}\right) \left[\sqrt{1 + (Ft/mc)^{2}} - 1\right].$$

This is a hyperbola. The classical answer is of course a parabola: $x = \frac{1}{2} (F/m) t^2$. Study-time question: Prove that for small times, the two results agree. (Show diagram.)

Relativity and Electrodynamics

Electromagnetic Force

Relativistically, the Lorentz force

$$\vec{F} = q\left(\vec{E} + \vec{u} \times \vec{B}\right)$$

F = quB

remains the same.

Cyclotron Motion

If the electric field is zero and the magnetic field is constant and uniform, the trajectory of a charged particle is a circle. Of course, the initial velocity must be perpendicular to the field. The magnetic force provides the centripetal force. Since $\vec{u} \perp \vec{B}$, the Lorentz force gives

nd
urse,
$$u$$

 q
 (3.37)
 B
 R
 (3.38)

From the momentum vector diagram, the triangle made up of the vectors $\vec{p} + d\vec{p}$, \vec{p} , and $d\vec{p}$ is similar to the triangle made up of two radii (*R*) and the arc length *ds*. We know that $ds = Rd\theta$. Also, the momentum triangle gives $dp = pd\theta$, since $|\vec{p}| = |\vec{p} + d\vec{p}| = p$.

Now, $F = \frac{dp}{dt} = p \frac{d\theta}{dt}$. Further, $\frac{ds}{dt} = R \frac{d\theta}{dt}$. But $\frac{ds}{dt} = u$, so $\frac{d\theta}{dt} = \frac{u}{R}$. It follows that

$$F = \frac{pu}{R}.$$
(3.39)

 $d\vec{p}$ $d\theta$ $\vec{p} + d\vec{p}$

(3.40)

 $\vec{p} + d\vec{p}$

Equating Eqs. (3.38) and (3.39) gives

This is the same as the expression obtained in classical physics except that here, p is the relativistic momentum.

p =

Magnetism and Relativity

(Adapted from Griffiths' Introduction to Electrodynamics)

We shall show that the existence of magnetism can be deduced from electrostatics and relativity. We will show that the magnetic force between a moving charge and a current-carrying wire can be found without the use of the laws of magnetism.

Consider a string of positive charges moving to the right at speed u. The charges are separate but close enough together that they can be regarded as a continuous line charge λ .² We now superimpose another "line" charge $-\lambda$ moving to the left with speed u. The net current is $I = 2\lambda u$ to the right. [See figure to the right, part (a). The figure was taken from Griffiths]



Further, at a distance r from the line charges, there is a point charge, q, moving with speed v to the right. All speeds are measured relative to an inertial frame S. In the frame S, there is no electrical force on q because the line charges produce no net charge.

Let us now look at the situation from the point of view of an observer in another inertial frame S' in which q is at rest. From the Lorentz transformation, the speeds of the line charges are

² λ = charge per unit length

$$u'_{\pm} = \frac{u \mp v}{1 \mp \frac{vu}{c^2}},$$
(3.41)

where u'_+ it the speed of the line charge moving to the right and u'_- is the speed of the line charge moving to the left. Hence, to the observer in S', the line charges will be moving with different speeds. The line charge moving to the left moves faster and hence has a greater length contraction. This means that the faster-moving line charge has a greater charge per unit length. We assume that charge is invariant. Thus, in S', our composite "wire" has a net negative charge. [See part (b) of figure.] This net negative charge will give rise to an electric field E' at the position of q, which feels an electric force F' = qE'. Since there is a force on q in S', there must also be a force on it in S, to be consistent with Postulate 1 of relativity. This force is nonelectrical since the wire is neutral in S. This is the *magnetic force*. To obtain this force quantitatively, one can make a transformation of F' from S' to S (force transformation equations follow from the Lorentz transformation equations and $\vec{F} = d\vec{p}/dt$. The force F obtained is

$$F = -\left(\frac{\mu_0 I}{2\pi r}\right) qv, \qquad (3.42)$$

where the term in parentheses is readily identified as the magnetic field due to a long, straight wire.