ABSTRACT

Name: Suzanne M. Riehl  Department: Mathematical Sciences

Title: Spectral Functions Associated with Sturm-Liouville and Dirac Equations

Major: Mathematical Sciences  Degree: Doctor of Philosophy

Approved by:  Date:

Dissertation Director

NORTHERN ILLINOIS UNIVERSITY
ABSTRACT

Two questions relating to the spectral functions associated with limit point differential equations are addressed. The equations are the second order Sturm-Liouville equation and the first order two-dimensional system known as the Dirac equation. For each equation, conditions on the coefficient functions are given so that the spectral function derivatives may be given explicitly in the form of a series in terms of the given functions. Also, for each equation, formulae relating the spectral functions for different values of the initial condition are displayed.
SPECTRAL FUNCTIONS ASSOCIATED WITH STURM-LIOUVILLE AND
DIRAC EQUATIONS

A DISSERTATION SUBMITTED TO THE GRADUATE SCHOOL
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICAL SCIENCES

BY
SUZANNE M. RIEHL
© 2001 Suzanne M. Riehl

DEKALB, ILLINOIS
MAY 2001
Certification: In accordance with departmental and Graduate School policies, this dissertation is accepted in partial fulfillment of degree requirements.

__________________________________________
Dissertation Director

__________________________________________
Date
ACKNOWLEDGMENTS

I would like to extend my thanks to the faculty, staff, and graduate students at the Department of Mathematical Sciences, Northern Illinois University, for challenging, supporting, and encouraging me throughout my studies. In particular, I greatly appreciate the guidance and humour of my advisor, Dr. Bernard Harris. I have learned much from him and have thoroughly enjoyed the process. I also gratefully acknowledge the help of my committee: Dr. Linda Sons, Dr. Qingkai Kong, and, from the Dublin Institute of Technology, Dr. Daphne Gilbert. Their comments and questions pulled me to a higher level of mathematical maturity. As I continue in mathematics, I aim to be as effective in communicating the beauty and power of mathematics as all those who have influenced me.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>1.1</td>
<td>Overview</td>
<td>1</td>
</tr>
<tr>
<td>1.2</td>
<td>Necessary Background Information</td>
<td>4</td>
</tr>
<tr>
<td>1.3</td>
<td>The Operator Viewpoint</td>
<td>8</td>
</tr>
<tr>
<td>2.</td>
<td>SPECTRAL FUNCTIONS ASSOCIATED WITH STURM-LIOUVILLE EQUATIONS WITH POTENTIALS OF WIGNER-VON NEUMANN TYPE</td>
<td>11</td>
</tr>
<tr>
<td>2.1</td>
<td>Introduction</td>
<td>11</td>
</tr>
<tr>
<td>2.2</td>
<td>The Results</td>
<td>12</td>
</tr>
<tr>
<td>2.3</td>
<td>The Riccati Equation</td>
<td>14</td>
</tr>
<tr>
<td>2.4</td>
<td>Proof of Theorem 2.1</td>
<td>17</td>
</tr>
<tr>
<td>2.5</td>
<td>An Example: The Wigner-von Neumann potential</td>
<td>19</td>
</tr>
<tr>
<td>3.</td>
<td>A REFINEMENT OF THE STURM-LIOUVILLE CONNECTION FORMULAE</td>
<td>23</td>
</tr>
<tr>
<td>3.1</td>
<td>Introduction</td>
<td>23</td>
</tr>
<tr>
<td>3.2</td>
<td>Proofs of the Connection Formulae</td>
<td>26</td>
</tr>
<tr>
<td>3.3</td>
<td>Examples</td>
<td>30</td>
</tr>
</tbody>
</table>
Chapter 4. CONNECTION FORMULAE FOR SPECTRAL FUNCTIONS ASSOCIATED WITH SINGULAR DIRAC EQUATIONS .... 33

4.1 Introduction .................................................. 33
4.2 The Connection Formulae ................................. 35
4.3 Proofs ............................................................. 37
4.4 Examples ...................................................... 40

Chapter 5. THE FORM OF THE SPECTRAL FUNCTIONS ASSOCIATED WITH DIRAC EQUATIONS .......... 42

5.1 Introduction .................................................. 42
5.2 The Representation of $\rho'_\alpha(\mu)$ ..................... 43
5.3 The Riccati Equation ....................................... 45
5.4 Proofs ............................................................. 50
5.5 Examination of Conditions for which Theorem 5.1 Holds .... 53
5.6 Examples ...................................................... 57

Chapter 6. A REFINEMENT OF THE DIRAC EQUATION CONNECTION FORMULA .................. 61

6.1 Introduction .................................................. 61
6.2 Proof of Theorem 6.1 ....................................... 63
6.3 Examples ...................................................... 64

REFERENCES ...................................................... 67
CHAPTER 1

INTRODUCTION

1.1 Overview

In this work, the spectral functions associated with two differential equations are studied. The first is the Sturm-Liouville equation

\[-(py')' + qy = \lambda wy\]  

(1.1.1)

on \([0, 1]\) together with the initial condition

\[y(0) \cos \alpha + p(0)y'(0) \sin \alpha = 0\]  

(1.1.2)

where \(\alpha \in [0, \pi]\). The functions \(p, q, \) and \(w\) are assumed to be real valued on \([0, \infty)\) with \(w(\cdot) > 0\) and \(p(\cdot) > 0\) and each of \(1/p, q, w \in L^1_{\text{loc}}[0, \infty)\). Some results will be given under more restrictive conditions. The complex spectral parameter is \(\lambda := \mu + i\epsilon, \mu, \epsilon \in \mathbb{R}\).

The second equation is the Dirac equation of the form

\[y' = \begin{pmatrix} p & \lambda + c + v_1 \\ -(\lambda - c + v_2) & -p \end{pmatrix} y, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\]  

(1.1.3)

on \([0, \infty)\) together with the initial condition

\[y_1(0) \cos \alpha + y_2(0) \sin \alpha = 0\]  

(1.1.4)
where \( \alpha \in [0, \pi) \). Also, \( c \geq 0 \) is a constant, \( \lambda = \mu + i\varepsilon \) is the complex spectral parameter, and \( v_1, v_2, \) and \( p \) are real valued members of \( L^1[0, \infty) \). Additional restrictions on the coefficient functions are imposed for some results.

These equations are related in the sense that equation (1.1.1) with \( p \equiv w \equiv 1 \) is a one-dimensional Schrödinger equation and equation (1.1.3) can be viewed physically as the radial component of the Dirac partial differential equation in relativistic quantum mechanics [10].

A spectral function \( \rho_\alpha(\mu) \) is associated with each equation and initial condition. This function is defined for \( \mu \in \mathbb{R} \) where \( \mu = \text{Re}\{\lambda}\). The spectral function is nondecreasing and can be thought of, roughly, as a probability distribution function. The symmetric derivative \( \rho'_\alpha(\mu) \) of the spectral function is defined by

\[
\rho'_\alpha(\mu) = \lim_{\varepsilon \to 0^+} \frac{\rho_\alpha(\mu + \varepsilon) - \rho_\alpha(\mu - \varepsilon)}{2\varepsilon}
\]

Two questions regarding these spectral functions are addressed. It is known that with appropriate conditions on the coefficient functions, the spectral functions associated with (1.1.1) can be given explicitly in the form of a series. Harris [11] considers (1.1.1) with \( p \equiv w \equiv 1 \) and proves the following.

**Theorem 1.1** A sequence of functions is defined as follows.

\[
v_1(x, \mu) = -\int_x^\infty \exp(2i\mu^{1/2}(t - x)) q(t) \, dt
\]

\[
v_{n+1}(x, \mu) = \int_x^\infty \exp \left(2i\mu^{1/2}(t - x) + \int_x^t \sum_{k=1}^n v_k(s, \mu) \, ds \right) v_n(t, \mu)^2 \, dt
\]

\[
v(x, \mu) = i\mu^{1/2} + \sum_{n=1}^\infty v_n(x, \mu)
\]

\[
S(x, \mu) = \text{Re}\{v(x, \mu)\}
\]
\[ T(x, \mu) = \text{Im}\{v(x, \mu)\} \]

If there exist functions \(a(x)\) and \(\eta(\mu)\) and a \(\mu_0 > 0\) such that

\[
\left| \int_x^\infty \exp(2i\mu^{1/2}t)q(t) \, dt \right| \leq a(x)\eta(\mu) \text{ for } \mu \geq \mu_0 \text{ and } 0 \leq x < \infty
\]

where \(a(x)\) is decreasing, \(a(\cdot) \in L^1[0, \infty)\), and \(\eta(\mu) \to 0\) as \(\mu \to \infty\), then

\[
\rho_\alpha'(\mu) = \frac{\pi^{-1} \mu^{-1/2} T(0, \mu)^2 \exp\left(-2 \int_0^\infty S(t, \mu) \, dt\right)}{(S(0, \mu)^2 + T(0, \mu)^2) \sin^2 \alpha + S(0, \mu) \sin 2\alpha + \cos^2 \alpha} \tag{1.1.5}
\]

for \(\alpha \in [0, \pi)\).

It is also shown that as \(\mu \to \infty\), \(\rho_\alpha'(\mu) \sim \mu^{1/2}/\pi\) and \(\rho_\alpha'(\mu) \sim \mu^{-1/2}/\pi\), for all \(\alpha \in (0, \pi)\). This is consistent with the asymptotic behavior of the spectral functions in [16]. Using a different sequence of functions \(\{v_n\}\), Gilbert and Harris [8] show the spectral derivative can be written

\[
\rho_\alpha'(\mu) = \frac{1}{\pi} \frac{T(0, \mu)}{(S(0, \mu)^2 + T(0, \mu)^2) \sin^2 \alpha + S(0, \mu) \sin 2\alpha + \cos^2 \alpha} \tag{1.1.6}
\]

where \(S(x, \mu)\) and \(T(x, \mu)\) are as defined above. A key feature is that the \(S\) and \(T\) functions are defined in terms of the potential \(q\) and can be identified with the real and imaginary parts, respectively, of \(m_0(\mu)\), the Titchmarsh-Weyl function for \(\alpha = 0\).

Theorem 1.1 holds, for example, if \(q\) is either decreasing on \([0, \infty)\) and a member of \(L^1[0, \infty)\) or if \((1 + t)q(t) \in L^1[0, \infty)\). In Chapter 2, this theorem is extended to include a different class of functions \(q\). Specifically, the spectral function is proved to have the form (1.1.6) for \(q\) which are conditionally integrable in a sense that includes the Wigner-von Neumann potentials.
In Chapter 5, with suitable conditions on the coefficient functions, a representation for spectral functions associated with the Dirac equation (1.1.3) is established. This representation has the same dependence on \( m_0(\mu) \) as (1.1.6) where the real and imaginary parts of \( m_0(\mu) \) are given in terms of the coefficient functions of the Dirac equation. The spectral functions for the Sturm-Liouville and Dirac equations, however, have distinctly different asymptotic behaviors.

The other central question addressed is the relationships among spectral functions for distinct initial conditions in a fixed equation. Gilbert and Harris answer this question for (1.1.1) in [7]. In Chapter 3, their result is refined for special cases of the equation. In Chapters 4 and 6, connection formulae for the Dirac equation are established. There are surprising similarities and differences in the connection formulae for the two equations.

### 1.2 Necessary Background Information

It is convenient to denote particular solutions of the Sturm-Liouville and Dirac equations as follows. We let \( \varphi_\alpha(x, \lambda) \) and \( \theta_\alpha(x, \lambda) \) be solutions of (1.1.1) satisfying

\[
\begin{align*}
\theta_\alpha(0, \lambda) &= \cos \alpha \\
p(0)\theta_\alpha'(0, \lambda) &= \sin \alpha
\end{align*}
\]

or let them be solutions of (1.1.3) satisfying

\[
\begin{align*}
\varphi_\alpha(0, \lambda) &= -\sin \alpha \\
p(0)\varphi_\alpha'(0, \lambda) &= \cos \alpha
\end{align*}
\]

where \( \alpha \in [0, \pi) \). \( \varphi_\alpha(x, \lambda) \) is a regular solution of the initial value problem as it satisfies the initial condition (1.1.2) or (1.1.4). Since \( \theta_\alpha \) and \( \varphi_\alpha \) are linearly independent
(their Wronskian is nonzero), every solution of the Sturm-Liouville or Dirac equation can be written as a linear combination of them.

It is well known that (1.1.1) and (1.1.3) fall into exactly one of two types [3, 12, 17]. If, for any \( \lambda \in \mathbb{C} \), every solution of the differential equation is a member of \( L^2[0, \infty) \), then the equation is called limit circle at infinity. Otherwise, the equation is called limit point at infinity. This classification is independent of \( \lambda \). Here we are concerned with limit point equations exclusively. In this case it is further known that for each nonreal \( \lambda \), there is exactly one solution, up to constant multiples, belonging to \( L^2[0, \infty) \). This solution is denoted \( \psi_\alpha(\cdot, \lambda) \). The Titchmarsh-Weyl function, \( m_\alpha(\lambda) \), is that function such that

\[
\theta_\alpha(x, \lambda) + m_\alpha(\lambda) \varphi_\alpha(x, \lambda) = \psi_\alpha(x, \lambda) \in L^2[0, \infty).
\] (1.2.3)

The function \( m_\alpha(\lambda) \) is a Herglotz (or Nevanlinna-Pick) function, that is, it is analytic in the upper half-plane and maps the upper half-plane to itself. \( m_\alpha(\lambda) \) is also analytic in the lower half-plane with negative imaginary part. In general, these functions may not be analytic continuations of each other. Any pole of \( m_\alpha(\lambda) \) is simple and on the real axis. \( m_\alpha(\lambda) \) can be represented in the form of a Cauchy-Stieltjes integral involving the spectral function \( \rho_\alpha(\mu) \). Additionally, \( m_\alpha(\lambda) \) and \( m_\beta(\lambda) \) are related by the connection formula

\[
m_\beta(\lambda) = \frac{m_\alpha(\lambda) \cos(\beta - \alpha) - \sin(\beta - \alpha)}{m_\alpha(\lambda) \sin(\beta - \alpha) + \cos(\beta - \alpha)}.
\] (1.2.4)

We refer to [3, 6, 9, 12, 17, 18, 20].

Conversely, subject to normalization, the spectral function \( \rho_\alpha(\mu) \) is uniquely determined by \( m_\alpha(\lambda) \). Since the spectral function is nondecreasing, it may be written
as the sum of nondecreasing functions

\[ \rho_\alpha = \rho_\alpha^{(ac)} + \rho_\alpha^{(sc)} + \rho_\alpha^{(p)} \]

where \( \rho_\alpha^{(ac)} \) is absolutely continuous, \( \rho_\alpha^{(sc)} \) is singularly continuous, and \( \rho_\alpha^{(p)} \) is a step function. This decomposition is unique up to the addition of constants. The spectrum of a problem is defined as points of nonconstancy of the spectral function. More precisely, it is the complement of the set of points in a neighborhood of which \( \rho_\alpha \) is constant \([6, 17]\). It is necessarily a closed set. The absolutely continuous spectrum consists of the nonconstancy points of \( \rho_\alpha^{(ac)} \); the singularly continuous and point spectra are defined similarly.

Formulae connecting \( \rho_\alpha(\lambda) \) and \( m_\alpha(z) \) are

\[ \rho_\alpha(\nu) - \rho_\alpha(\mu) = \lim_{\epsilon \to 0^+} \frac{1}{\pi} \int_{\mu}^{\nu} \text{Im}\{m_\alpha(x + i\epsilon)\} \, dx \quad (1.2.5) \]

for all \( \mu, \nu \in \mathbb{R} \) which are points of continuity of \( \rho_\alpha(\lambda) \) and

\[ m_\alpha(z) = \int_{-\infty}^{\infty} \frac{1}{\lambda - z} \, d\rho_\alpha(\lambda) + \cot \alpha \quad (1.2.6) \]

which holds for all \( \alpha \in (0, \pi) \).

This results of this work concern the absolutely continuous spectra associated with equations \((1.1.1)\) and \((1.1.3)\). The following version of \((1.2.5)\) referred to as the Titchmarsh-Kodaira formula supplies the connection between \( m_\alpha(\lambda) \) and \( \rho_\alpha(\mu) \) :

\[ \rho_\alpha'(\mu) = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \text{Im}\{m_\alpha(\mu + i\epsilon)\} \quad (1.2.7) \]

where the limit exists. See \([3, 12, 17]\).
Gilbert and Harris [7] prove the following results for Sturm-Liouville limit point equations (1.1.1).

**Theorem 1.2** For almost all \( \mu \in \mathbb{R} \), if there is an \( \alpha \in [0, \pi) \) such that \( 0 < \rho'_\alpha(\mu) < \infty \), then we have the following.

(i) \( \rho'_\beta(\mu) \) exists, with \( 0 < \rho'_\beta(\mu) < \infty \) for all \( \beta \in [0, \pi) \).

(ii) For all \( \alpha_1, \alpha_2 \in (0, \pi) \backslash \{\pi/2\} \),

\[
2\rho'_0(\mu) \left\{ \frac{1}{\rho'_{\alpha_1}(\mu) \sin 2\alpha_1} - \frac{1}{\rho'_{\alpha_2}(\mu) \sin 2\alpha_2} \right\} = \frac{\rho'_0(\mu)}{\rho'_{\pi/2}(\mu)} \{\tan \alpha_1 - \tan \alpha_2\} + \cot \alpha_1 - \cot \alpha_2. \tag{1.2.8}
\]

**Theorem 1.3** For almost all \( \mu \in \mathbb{R} \), if there is an \( \alpha \in [0, \pi) \) such that \( 0 < \rho'_\alpha(\mu) < \infty \), and if \( \alpha, \alpha + \pi/2 \) and \( \beta \) are distinct members of \( [0, \pi) \), then

\[
\frac{\rho'_\alpha(\mu)}{\rho'_{\alpha+\pi/2}(\mu)} - (\pi \rho'_\alpha(\mu))^2 = \frac{1}{\sin^2(2\beta - 2\alpha)} \left\{ \frac{\rho'_\alpha(\mu)}{\rho'_\beta(\mu)} - \frac{\rho'_\alpha(\mu)}{\rho'_{\alpha+\pi/2}(\mu)} \sin^2(\beta - \alpha) - \cos^2(\beta - \alpha) \right\}^2. \tag{1.2.9}
\]

Eastham [4] rewrites (1.2.8) for arbitrary \( \alpha, \beta, \gamma, \delta \in [0, \pi) \):

\[
\frac{\sin(\beta - \gamma) \sin(\gamma - \delta) \sin(\delta - \beta)}{\rho'_\alpha(\mu)} - \frac{\sin(\gamma - \delta) \sin(\delta - \alpha) \sin(\alpha - \gamma)}{\rho'_\beta(\mu)} + \frac{\sin(\delta - \alpha) \sin(\alpha - \beta) \sin(\beta - \delta)}{\rho'_\gamma(\mu)} - \frac{\sin(\alpha - \beta) \sin(\beta - \gamma) \sin(\gamma - \alpha)}{\rho'_\delta(\mu)} = 0.
\]

These theorems display the relations among the spectral derivatives. For this general setting, knowledge of \( \rho'_\alpha(\mu) \) for three distinct \( \alpha \) is sufficient to uniquely determine any fourth spectral derivative. Also, (1.2.9) relates the spectral derivatives
for just three distinct initial conditions. In Chapter 3, this result is written for three arbitrary initial conditions and, for special cases of (1.1.1), a third spectral derivative is uniquely determined.

We fix notation. For the complex number \( z = re^{i\theta} \), \( \text{Arg}\{z\} \) represents the argument of \( z \) where \( \theta \in [0, 2\pi) \) and \( \sqrt{z} \) is taken as \( \sqrt{re^{i\theta/2}} \).

1.3 The Operator Viewpoint

Spectral functions can also be approached from the point of view of the general spectral theory of linear selfadjoint operators in a Hilbert space with scalar product \( (f, g) \).

Consider a set \( D_T \) of elements of a Hilbert space \( H \). If to each \( f \in D_T \) there is associated some element \( Tf \in H \), then \( T \) is called an operator in \( H \) with domain \( D_T \). \( T \) is a linear operator if \( T \) satisfies \( T(\alpha f + \beta g) = \alpha Tf + \beta Tg \) for any \( \alpha, \beta \in \mathbb{C} \) and any \( f, g \in D_T \). An operator \( T \) is closed if from the existence of the limits

\[
\lim_{n \to \infty} f_n = f, \quad \lim_{n \to \infty} Tf_n = g, \quad \text{with} \quad f_n \in D_T,
\]

it follows that \( f \in D_T \) and \( g = Tf \). Now there are pairs of elements \( g, g^* \) for which \( (Tf, g) = (f, g^*) \) holds for every \( f \in D_T \). If \( D_T \) is dense in \( H \), then the element \( g^* \) is uniquely determined by the element \( g \). \( T^* \), the adjoint of \( T \), is that operator with \( g^* = T^*g \). An operator \( T \) is selfadjoint if \( T = T^* \), that is, \( D_T = D_{T^*} \) and \( Tf = T^*f \) for all \( f \in D_T \). A few of the many properties of selfadjoint operators are as follows. First, if a selfadjoint operator has an inverse, then this inverse is selfadjoint. The eigenvalues of a selfadjoint operator are real. Also, two eigenvectors \( f_1, f_2 \) of a selfadjoint operator corresponding to distinct eigenvalues \( \lambda_1, \lambda_2 \) are orthogonal.

Let \( T \) be a closed linear operator defined on \( D_T \), a dense subset of \( H \). If \( T - \lambda I \)
is invertible (that is, the operator \((T - \lambda I)^{-1}\) exists, is defined everywhere in \(H\), and is bounded), then \(\lambda\) is said to be in the resolvent set of \(T\) and is known as a regular point of the operator \(T\). Otherwise, \(\lambda\) is in the spectrum of \(T\), denoted \(\sigma(T)\). For \(T\) a selfadjoint operator, \(\sigma(T) \subset \mathbb{R}\). The spectrum can be decomposed into the point spectrum, the singularly continuous spectrum, and the absolutely continuous spectrum.

For the Sturm-Liouville case, we consider the differential expression \(L\) defined by

\[
Ly = \frac{1}{w}(-(py')' + qy).
\]

Then equation (1.1.1) is equivalent to \(Ly = \lambda y\). The relevant Hilbert space is

\[
L^2[0, \infty; w) = \left\{ f \mid \int_0^\infty |f(x)|^2w(x) \, dx < \infty \right\}
\]

with scalar product

\[
(f(x), g(x)) = \int_0^\infty f(x)\overline{g(x)}w(x) \, dx.
\]

The operator \(H_\alpha\) is defined by

\[
H_\alpha y = Ly \quad \text{for} \ y \in D
\]

where \(D\) is the set of functions \(y\) satisfying

i. \(y, py' \in L^1_{loc}[0, \infty)\)

ii. \(y, H_\alpha y \in L^2[0, \infty; w)\)

iii \(y(0) \cos \alpha + y'(0) \sin \alpha = 0\) for some fixed \(\alpha\) in \([0, \pi)\).
If $L$ is regular at 0 and limit point at infinity, then the operator $H_\alpha$ is selfadjoint for each $\alpha$ in $[0, \pi)$.

The spectral theorem applies to selfadjoint operators. This implies that there exists an associated non-decreasing spectral function from which the spectral properties of the operator can be determined.

A selfadjoint operator for the Dirac equation may be defined mutatis mutandis. We refer to [3, 6, 9, 12, 17, 21] and the references listed therein.
CHAPTER 2

SPECTRAL FUNCTIONS ASSOCIATED WITH STURM-LIOUVILLE EQUATIONS WITH POTENTIALS OF WIGNER-VON NEUMANN TYPE

2.1 Introduction

Let $\rho_\alpha(\mu)$ denote the spectral function associated with the equation

$$y'' + (\lambda - q)y = 0$$

(2.1.1)

on $[0, \infty)$ with the initial condition

$$y(0) \cos \alpha + y'(0) \sin \alpha = 0$$

(2.1.2)

where $\alpha \in [0, \pi)$ and $q$ is real-valued. We suppose $q$ is such that (2.1.1) is in the limit point case at infinity so that $\rho_\alpha(\mu)$ is unique, up to the addition of a constant function. As is usual, we normalize by taking $\rho_\alpha(0) = 0$. If $q \in L^1[0, \infty)$, the spectral function is known to be absolutely continuous for $\mu > 0$, [20]. Under quite weak extra conditions on $q$, the spectral function is calculated in [8] and [11] in the form of a series. If $q$ is conditionally integrable but not $L^1[0, \infty)$, the results of [8] and [11] do not apply. For certain of these $q$, however, Behncke [1, 2] shows that (2.1.1) is limit point at infinity and has absolutely continuous spectrum on the positive real line away from the set of resonance points. An example of such a $q$ is furnished by
the Wigner-von Neumann potential

\[ q(t) = \frac{A}{t + 1} \sin(2ct). \tag{2.1.3} \]

The purpose of the present chapter is to extend the results of [11] to cover potentials of this type and thus to give a series representation for the spectral derivative \( \rho'_\alpha(\mu) \) for \( \mu \geq \Lambda_0 \), where \( \Lambda_0 \) is computable. In §2.5, we present \( \rho'_\alpha(\mu) \) in the case where \( q \) is given by (2.1.3).

We note that asymptotic representations for spectral functions associated with (2.1.1), (2.1.2) are developed for the case when \( q \not\in L^1[0, \infty) \) but is small at infinity, in [5] and the earlier papers cited therein. These representations require \( q' \in L^1[0, \infty) \) and do not appear to be applicable to the case when \( q \) is given by (2.1.3). In this work, the oscillation of the conditionally convergent \( q \) is utilized in order to represent the spectral functions.

## 2.2 The Results

We write \( \lambda =: \mu + i\epsilon \) and \( \lambda^{1/2} =: \gamma + i\delta \) where \( \mu, \epsilon, \gamma, \delta \in \mathbb{R} \) and the principal branch of the square root is used. We take \( \mu, \gamma > 0 \) and \( \delta \geq 0 \) which implies that \( \epsilon \geq 0 \). We set

\[ v_1(x, \lambda) := -\int_x^\infty e^{2i\lambda^{1/2}(t-x)} q(t) \, dt \tag{2.2.1} \]

\[ v_2(x, \lambda) := \int_x^\infty e^{2i\lambda^{1/2}(t-x)} v_1(t, \lambda)^2 \, dt \tag{2.2.2} \]

and for \( n \geq 3 \),

\[ v_n(x, \lambda) := \int_x^\infty e^{2i\lambda^{1/2}(t-x)} \left( v_{n-1}(t, \lambda)^2 + 2v_{n-1}(t, \lambda) \sum_{m=1}^{n-2} v_m(t, \lambda) \right) \, dt. \tag{2.2.3} \]
We further write

\[ S(t, \lambda) + iT(t, \lambda) := i\lambda^{1/2} + \sum_{n=1}^{\infty} v_n(t, \lambda) \]  

(2.2.4)

where \( S \) and \( T \) are real-valued.

The conditions imposed on \( q \) throughout this chapter are referred to as conditions (Q) and are as follows. We suppose that there exist functions \( a, b, \eta_1, \eta_2 \) so that

(i) \( |v_1(x, \lambda)| \leq a(x)\eta_1(\lambda) \), where \( \eta_1(\lambda) \to 0 \) as \( |\lambda| \to \infty \) and \( a(x) \) decreases to 0 as \( x \to \infty \).

(ii) \( |v_2(x, \lambda)| \leq b(x)\eta_2(\lambda) \), where \( \eta_2(\lambda) \to 0 \) as \( |\lambda| \to \infty \) and \( b(x) \) is a decreasing \( L^1[0, \infty) \) function.

(iii) There exists \( K > 0 \) so that \( \int_{x}^{\infty} a(t)b(t) \, dt \leq \frac{K}{4}b(x) \) for \( x \geq 0 \).

(iv) \( \left| \int_0^x v_1(t, \lambda) \, dt \right| < \infty \) for \( 0 \leq x < \infty \).

**Theorem 2.1** If \( q \) satisfies conditions (Q) and if \( \Lambda_0 \in \mathbb{R} \) is so large that

\[ 9\eta_2(\lambda) \int_0^{\infty} b(t) \, dt + K\eta_1(\lambda) \leq 1 \]

for all \( \lambda = \mu + i\epsilon \) with \( \mu \geq \Lambda_0 \), then for all \( \mu \geq \Lambda_0 \)

\[ \rho'_0(\mu) = \frac{1}{\pi} T(0, \mu) \]

and, for \( \alpha \in (0, \pi) \),

\[ \rho'_\alpha(\mu) = \frac{1}{\pi} \frac{T(0, \mu)}{(S(0, \mu)^2 + T(0, \mu)^2) \sin^2 \alpha + S(0, \mu) \sin 2\alpha + \cos^2 \alpha}. \]

We note here that the proof will show \( 0 < \rho'_\alpha(\mu) < \infty \) for all \( \alpha \in (0, \pi) \) and \( \mu \geq \Lambda_0 \) and that \( S(\mu) \to 0 \) and \( T(\mu) \to \mu^{1/2} \) as \( \mu \to \infty \). The theorem is proved in §2.4 after the groundwork is laid in §2.3. An immediate consequence is as follows.
**Corollary 2.2** \( \Lambda_0 \) is an upper bound for the set of resonances.

**Proof.** A resonance point is an eigenvalue embedded in the continuous spectrum and the spectral derivative does not exist as finite number at such a point. But since \( 0 < \rho'_{\alpha}(\mu) < \infty \) for all \( \alpha \in [0, \pi] \) and \( \mu \geq \Lambda_0 \), it follows that \( \Lambda_0 \) is an upper bound for the set of resonances. Q.E.D.

The following representation of \( v_2(x, \lambda) \) is useful in applications of the theorem and will be proved in §2.4. It is also possible to represent each \( v_n(x, \lambda) \) in an analogous manner.

\[
v_2(x, \lambda) = \frac{v_1(x, \lambda)^2}{2i\lambda^{1/2}} + \frac{1}{i\lambda^{1/2}} \int_{x}^{\infty} e^{2i\lambda^{1/2}(t-x)}v_1(t, \lambda)q(t) \, dt, \quad x \geq 0. \tag{2.2.5}
\]

**2.3 The Riccati Equation**

Equation (2.1.1) is assumed to be in the limit point case at infinity which implies that for \( \text{Im}\{\lambda\} > 0 \), there is a \( L^2[0, \infty) \) solution \( \psi(x, \lambda) \), unique up to constant multiples. We denote by \( \theta_\alpha(\cdot, \lambda) \) and \( \varphi_\alpha(\cdot, \lambda) \) the solutions of (2.1.1) satisfying the initial conditions

\[
\begin{align*}
\theta_\alpha(0, \lambda) &= \cos \alpha & \varphi_\alpha(0, \lambda) &= -\sin \alpha \\
\theta'_\alpha(0, \lambda) &= \sin \alpha & \varphi'_\alpha(0, \lambda) &= \cos \alpha.
\end{align*}
\tag{2.3.1}
\]

These solutions, together with \( \psi(x, \lambda) \), enable us to define the Titchmarsh-Weyl \( m \)-function by

\[
\psi(x, \lambda) = \theta_\alpha(x, \lambda) + m_\alpha(\lambda)\varphi_\alpha(x, \lambda).
\tag{2.3.2}
\]

It is known, [3, 12, 20], that \( m_\alpha(\lambda) \) is well defined, nonzero, and analytic for \( \text{Im}\{\lambda\} > 0 \). Further, if \( \text{Im}\{\lambda\} > 0 \), \( \psi(x, \lambda) \neq 0 \) for any \( x \in [0, \infty) \); this follows from the limit
point construction (see, for example, [3, chapter 9]). The ratio \( \psi'(x, \lambda)/\psi(x, \lambda) \) is unique and by equating representations of \( \psi'(0, \lambda)/\psi(0, \lambda) \) for distinct values of the initial value parameter, we obtain the connection formula

\[
m_{\alpha}(\lambda) = \frac{m_{\beta}(\lambda) \cos(\alpha - \beta) - \sin(\alpha - \beta)}{m_{\beta}(\lambda) \sin(\alpha - \beta) + \cos(\alpha - \beta)}.
\] (2.3.3)

The spectral derivative, \( \rho'_\alpha(\mu) \), may be obtained from the Titchmarsh-Kodaira formula (1.2.7) wherever \( \lim_{\varepsilon \to 0} m_{\alpha}(\mu + i\varepsilon) \) exists.

Let \( y(x, \lambda) \) denote a solution of (2.1.1) which does not vanish for \( x \in [0, \infty) \) and denote

\[
v(x, \lambda) := \frac{y'(x, \lambda)}{y(x, \lambda)}.
\]

It may readily be shown that \( v(x, \lambda) \) satisfies the Riccati equation

\[
v' = -\lambda + q - v^2
\] (2.3.4)

for \( x \in [0, \infty) \). We seek a solution of (2.3.4) of the form

\[
v(x, \lambda) = i\lambda^{1/2} + \sum_{n=1}^{\infty} v_n(x, \lambda)
\] (2.3.5)

where \( \lim_{x \to \infty} v(x, \lambda) = i\lambda^{1/2} \). Following the analysis of [8] we substitute (2.3.5) into (2.3.4) and obtain

\[
v'(x, \lambda) = \sum_{n=1}^{\infty} (v'_n + 2i\lambda^{1/2}v_n) = q - v_1^2 - \sum_{n=2}^{\infty} (v_n^2 + 2v_n \sum_{m=1}^{n-1} v_m).
\] (2.3.6)

The sum may be decomposed as

\[
v'_1 + 2i\lambda^{1/2}v_1 = q \quad (2.3.7)
\]
\[
v'_2 + 2i\lambda^{1/2}v_2 = -v_1^2 \quad (2.3.8)
\]
\[ v'_n + 2i\lambda^{1/2}v_n = -\left(v_{n-1}^2 + 2v_{n-1}\sum_{m=1}^{n-2}v_m\right), \quad n \geq 3. \tag{2.3.9} \]

These equations are satisfied by the \( \{v_n(x, \lambda)\} \) of (2.2.1)–(2.2.3). The term-by-term differentiation and rearrangement of \( v(x, \lambda) \) are justified by the absolute uniform convergence of (2.3.5) which is proved in the following lemma.

**Lemma 2.3** If \( q \) satisfies the conditions (Q) for \( 0 \leq x < \infty \) and \( \text{Im}\{\lambda\} \geq 0 \) then there exists \( \Lambda_0 \in \mathbb{R} \) such that for all \( \lambda = \mu + i\varepsilon \) with \( \mu \geq \Lambda_0 \) and \( n \geq 2 \)

\[ |v_n(x, \lambda)| \leq \frac{\eta_2(\lambda)|b(x)|}{2^{n-2}} \text{ for } 0 \leq x < \infty. \]

**Proof.** We use induction on \( n \). The result is clear for \( n = 2 \). Suppose it were true for \( n = 2, \ldots, k; \) then

\[
|v_{k+1}(x, \lambda)| \leq \int_x^{\infty} \left(|v_k|^2 + 2|v_k||v_1| + 2|v_k|\sum_{m=2}^{k-1}|v_m|\right) dt
\]

\[
\leq \int_x^{\infty} \left(\frac{\eta_2(\lambda)^2b(t)^2}{2^{2k-4}} + \frac{2\eta_2(\lambda)\eta_1(\lambda)a(t)b(t)}{2^{k-2}} + \frac{2\eta_2(\lambda)^2b(t)^2}{2^{k-2}}\sum_{m=2}^{\infty}\frac{1}{2^{m-2}}\right) dt
\]

\[
\leq \frac{\eta_2(\lambda)}{2^{k-1}} \left[4\eta_1(\lambda)\int_{x}^{\infty} a(t)b(t) \, dt + \left(8 + \frac{1}{2^{k-3}}\right)\eta_2(\lambda)\int_{x}^{\infty} b(t)^2 \, dt\right]
\]

\[
\leq \frac{\eta_2(\lambda)b(x)}{2^{k-1}} \left[K\eta_1(\lambda) + 9\eta_2(\lambda)\int_{0}^{\infty} b(t) \, dt\right].
\]

The result now follows if \( \Lambda_0 \) is so large that the bracketed term is less than or equal to 1 for all \( \lambda \) with \( \mu \geq \Lambda_0 \). Q.E.D.

Lemma 2.3 implies that \( \sum_{n=1}^{\infty} v_n(x, \lambda) \) is uniformly absolutely convergent and hence that \( v(x, \lambda) = i\lambda^{1/2} + \sum_{n=1}^{\infty} v_n(x, \lambda) \) is continuous in \( x \) and \( \lambda \) for \( \text{Re}\{\lambda\} \geq \Lambda_0, \text{Im}\{\lambda\} \geq \)
0, and \( x \in [0, \infty) \). Also, from (2.3.9) for \( n \geq 3 \),

\[
v'_n = -2i\lambda^{1/2}v_n - v_{n-1}^2 - 2v_{n-1} \sum_{m=1}^{\infty} v_m
\]

so

\[
|v'_n| \leq \frac{2|\lambda^{1/2}|\eta_2(\lambda)b(x)}{2n-2} + \frac{\eta_2(\lambda)^2b(x)^2}{2n-6} + \frac{4\eta_2(\lambda)^2b(x)^2}{2n-3} + \frac{2\eta_1(\lambda)\eta_2(\lambda)a(x)b(x)}{2n-3}
\]

\[
\leq \frac{\eta_2(\lambda)b(x)}{2n-3} \left[ |\lambda^{1/2}| + \frac{\eta_2(\lambda)b(x)}{2n-3} + 4\eta_2(\lambda)b(x) + 2\eta_1(\lambda)a(x) \right].
\]

For each \( \lambda \), the expression in brackets is bounded. A similar calculation shows \( |v'_1|, |v'_2| \) are bounded for each \( \lambda \). Hence \( v(x, \lambda) \) is differentiable and therefore indeed represents a solution to (2.3.4) for \( x \in [0, \infty) \), \( Im\{\lambda\} \geq 0 \) and \( Re\{\lambda\} \geq \Lambda_0 \) where \( \Lambda_0 \) is computed as in Lemma 2.3.

### 2.4 Proof of Theorem 2.1

We show first that \( v(x, \lambda) \), the solution of (2.3.5) with \( \lim_{x \to -\infty} v(x, \lambda) = i\lambda^{1/2} \), is equal to \( \frac{\psi'(x, \lambda)}{\psi(x, \lambda)} \) where \( \psi(x, \lambda) \) is the \( L^2[0, \infty) \) solution of (2.1.1) for \( Im\{\lambda\} > 0 \). It is clear that a solution of (2.1.1) which does not vanish for \( x \in [0, \infty) \) is represented by

\[
(constant) \ exp \left\{ \int_0^x v(t, \lambda) \ dt \right\}.
\]

To prove that this solution is in \( L^2[0, \infty) \), we show that for \( Im\{\lambda\} > 0 \) there exists \( X \) with

\[
Re \left\{ \int_0^x v(t, \lambda) \ dt \right\} < -\frac{\delta}{2} x \quad \text{for} \quad x \geq X
\]
where, in the notation of §2.2, \( \delta = Im\{\lambda^{1/2}\} \). We note from (2.3.5) and Lemma 2.3 that

\[
Re \left\{ \int_0^x v(t, \lambda) \, dt \right\} = -\delta x + Re \left\{ \int_0^x v_1(t, \lambda) \, dt \right\} + Re \left\{ \int_0^x \sum_{n=2}^{\infty} v_n(t, \lambda) \, dt \right\} \\
\leq -\delta x + \left| \int_0^x v_1(t, \lambda) \, dt \right| + \sum_{n=2}^{\infty} \int_0^x |v_n(t, \lambda)| \, dt \\
\leq -\delta x + \left| \int_0^x v_1(t, \lambda) \, dt \right| + 2\eta_2(\lambda) \int_0^\infty b(t) \, dt.
\]

If \( X \) is so large that

\[
\sup_{x \in [X, \infty)} \left| \int_0^x v_1(t, \lambda) \, dt \right| + 2\eta_2(\lambda) \int_0^\infty b(t) \, dt < \frac{\delta x}{2},
\] (2.4.1)

then the result follows. Such an \( X \) must exist by conditions (Q) since the left-hand side of (2.4.1) is bounded while, for any \( \delta > 0 \), the right hand side goes to infinity with \( x \). Since the \( L^2[0, \infty) \) solution is unique (up to constant multiples), we conclude \( \psi(x, \lambda) = (\text{constant}) \exp \left\{ \int_0^x v(t, \lambda) \, dt \right\} \), and hence that \( v(x, \lambda) = \psi'(x, \lambda)/\psi(x, \lambda) \).

For \( \alpha = 0 \), (2.3.1) and (2.3.2) show that

\[
v(0, \lambda) = \frac{\psi'(0, \lambda)}{\psi(0, \lambda)} = m_0(\lambda) \text{ for } Im(\lambda) > 0, \mu \geq \Lambda_0.
\]

The solution \( v(x, \lambda) \), and \( v(0, \lambda) \) in particular, is defined for \( Im(\lambda) = 0, \mu \geq \Lambda_0 \).

Thus \( \lim_{\epsilon \to 0^+} m_0(\mu + i\epsilon) \) exists and the Titchmarsh-Kodaira formula (1.2.7) and (2.2.4) yield

\[
\rho'_0(\mu) = \frac{1}{\pi} \lim_{\epsilon \to 0^+} Im\{m_0(\mu + i\epsilon)\} = \frac{1}{\pi} T(0, \mu).
\]

Equation (2.3.3) with \( \beta = 0 \) now gives

\[
\rho'_\alpha(\mu) = \frac{T(0, \mu)}{\pi (S(0, \mu)^2 + T(0, \mu)^2) \sin^2 \alpha + S(0, \mu) \sin(2\alpha) + \cos^2 \alpha}
\]
and the theorem is proved. Q.E.D.

To verify (2.2.5), the alternate representation of \( v_2(x, \lambda) \), we use integration by parts together with (2.2.2) and (2.3.8) to see that

\[
v_2(x, \lambda) = \frac{v_1(t, \lambda)^2 e^{2i\lambda^{1/2}(t-x)}}{2i\lambda^{1/2}} \left| \right. _{x}^{\infty} - \frac{1}{i\lambda^{1/2}} \int_{x}^{\infty} e^{2i\lambda^{1/2}(t-x)} v_1(t, \lambda) v'_1(t, \lambda) \, dt
\]

\[
= - \frac{v_1(x, \lambda)^2}{2i\lambda^{1/2}} - \frac{1}{i\lambda^{1/2}} \int_{x}^{\infty} e^{2i\lambda^{1/2}(t-x)} v_1(t, \lambda)(q(t) - 2i\lambda^{1/2} v_1(t, \lambda)) \, dt
\]

\[
= - \frac{v_1(x, \lambda)^2}{2i\lambda^{1/2}} + 2v_2(x, \lambda) - \frac{1}{i\lambda^{1/2}} \int_{x}^{\infty} e^{2i\lambda^{1/2}(t-x)} v_1(t, \lambda) q(t) \, dt.
\]

The result now follows. A similar technique may be used to exhibit corresponding formulae for the other \( \{v_n(x, \lambda)\} \).

### 2.5 An Example: The Wigner-von Neumann potential

We show now that

\[
q(t) = \frac{A}{t+1} \sin(2ct), \quad c > 0, \ A \in \mathbb{R}
\]

(2.5.1)

satisfies the hypotheses of Theorem 2.1. The next lemma is useful in integrations involving this \( q \).

**Lemma 2.4** If \( \beta > 0, k \in \mathbb{R} \) where \( k \) and \( \lambda^{1/2} \) are not both 0, and \( \operatorname{Im}\{\lambda\} \geq 0 \), then

\[
\int_{x}^{\infty} e^{2i\lambda^{1/2}(t-x)} (t+1)^{-\beta} e^{ikt} \, dt = \frac{-(x+1)^{-\beta} e^{ikx}}{i(2\lambda^{1/2} + k)} + E(x, \lambda, \beta, k)
\]

where

\[
|E(x, \lambda, \beta, k)| \leq \frac{2\beta(x+1)^{-\beta-1}}{(2|\lambda|^{1/2} - |k|)^2}.
\]
Proof. This follows readily after integration by parts. Q.E.D.

Note also that

$$|E(x, \lambda, \beta, k)| = |E(x, \lambda, \beta, -k)|$$ (2.5.2)

For \( q \) as in (2.5.1), \( v_1(x, \lambda) \) is written

$$v_1(x, \lambda) = -\frac{A}{2i} \int_x^\infty e^{2i\lambda^{1/2}(t-x)}(t+1)^{-1}(e^{2ict} - e^{-2ict}) \, dt.$$ (2.5.3)

We apply Lemma 2.4 and (2.5.2) to (2.5.3) which gives

$$v_1(x, \lambda) = -\frac{A(x+1)^{-1}}{4} \left( \frac{e^{2icx}}{\lambda^{1/2} + c} - \frac{e^{-2icx}}{\lambda^{1/2} - c} \right)$$

$$- \frac{A}{2i} \left( E(x, \lambda, 1, 2c) - E(x, \lambda, 1, -2c) \right).$$ (2.5.4)

Thus,

$$|v_1(x, \lambda)| \leq \frac{|A|(x+1)^{-1}}{4} \left( \frac{2}{|\lambda|^{1/2} - c} \right) + \frac{|A|}{2} \frac{(x+1)^{-2}}{(|\lambda|^{1/2} - c)^2}$$

$$\leq \frac{|A|(x+1)^{-1}}{|\lambda^{1/2} - c|}$$

and we take \( a(x) := |A|(x+1)^{-1}, \eta_1(\lambda) := (|\lambda|^{1/2} - c)^{-1} \). Condition (i) of (Q) is satisfied.

Integration by parts also shows that

$$\left| \int_0^x v_1(t, \lambda) \, dt \right| < \infty \ \text{for} \ 0 \leq x < \infty$$

and so condition (iv) of (Q) is satisfied. To verify that \( v_2(x, \lambda) \) satisfies condition (ii) of (Q) we use (2.5.4) in (2.2.5). The first term of (2.2.5) is now

$$\frac{v_1(x, \lambda)^2}{2i\lambda^{1/2}} = \frac{A^2(x+1)^{-2}}{32i\lambda^{1/2}} \left( \frac{e^{4icx}}{(\lambda^{1/2} + c)^2} + \frac{e^{-4icx}}{(\lambda^{1/2} - c)^2} - \frac{2}{\lambda - c^2} \right)$$

$$+ \frac{A(x+1)^{-1}E_1(x, \lambda)}{4i\lambda^{1/2}} \left( \frac{e^{2icx}}{\lambda^{1/2} + c} - \frac{e^{-2icx}}{\lambda^{1/2} - c} \right) + \frac{E_1(x, \lambda)^2}{2i\lambda^{1/2}}$$ (2.5.5)
where
\[ E_1(x, \lambda) := \frac{A}{2i} \left( E(x, \lambda, 1, 2c) - E(x, \lambda, 1, -2c) \right) \]
 thus
\[ |E_1(x, \lambda)| \leq \frac{|A|(x + 1)^{-2}}{2(|\lambda|^{1/2} - c)^2}. \]

The second term of (2.2.5) is
\[
\frac{1}{i\lambda^{1/2}} \int_x^\infty e^{2i\lambda^{1/2}(t-x)} v_1(t, \lambda)q(t) \, dt =
\frac{A^2}{8\lambda^{1/2}} \int_x^\infty e^{2i\lambda^{1/2}t - x} (t+1)^{-2} \left( \frac{e^{4ict}}{\lambda^{1/2} + c} + \frac{e^{-4ict}}{\lambda^{1/2} - c} - \frac{1}{\lambda^{1/2} + c} - \frac{1}{\lambda^{1/2} - c} \right) dt
\]
\[
+ \frac{A}{2\lambda^{1/2}} \int_x^\infty e^{2i\lambda^{1/2}(t-x)} (t+1)^{-1} \left( e^{2ict} - e^{-2ict} \right) E_1(t, \lambda) \, dt.
\]

Some calculation then shows that
\[
|v_2(x, \lambda)| \leq \frac{|A|^2(x + 1)^{-2}}{2|\lambda|^{1/2}(|\lambda|^{1/2} - 2c)} + \frac{7|A|^2(x + 1)^{-3}}{8|\lambda|^{1/2}(|\lambda|^{1/2} - c)(|\lambda|^{1/2} - 2c)^2}
\]
\[
\leq \frac{2|A|^2(x + 1)^{-2}}{|\lambda|^{1/2}(|\lambda|^{1/2} - 2c)^2} \quad (2.5.6)
\]

We have supposed, in addition to the considerations of Theorem 2.1, that \(|\lambda|^{1/2} - c > 1\) and \(|\lambda|^{1/2} > 2c\).

It follows from (2.5.6) that \(|v_2(x, \lambda)| \leq b(x)\eta_2(\lambda)\) where
\[ b(x) := 2|A|^2(x + 1)^{-2} \text{ and } \eta_2(\lambda) := \frac{1}{|\lambda|^{1/2}(|\lambda|^{1/2} - 2c)^2}. \]

We note that condition (iii) of (Q) is met since
\[ \int_x^\infty a(t)b(t) \, dt = \int_x^\infty 2|A|^3(t+1)^{-3} \, dt = |A|^3(x + 1)^{-2} = \frac{|A|^2}{2} b(x). \]

Finally \(\Lambda_0\) can be taken to be the maximum of \((18A^2 + 2|A| + c)^2\) and \((2c + 1)^2\). This estimate is rather crude. For example, if \(A = c = 1\), then \(\Lambda_0 = 21^2\). However, using
the values of $A$ and $c$ in the calculations above, $\Lambda_0 \approx 23$. Note that in this case, $\lambda = 1$ is a resonance point and $\rho'_\alpha(1)$ does not exist. (The integral for $v(x,\lambda)$ is divergent for $\lambda^{1/2} = c = 1$.) But we have shown that the hypotheses of Theorem 2.1 are fulfilled by the Wigner-von Neumann potential (2.5.1) for $\mu > 23$ and hence the representation of the spectral function derivative follows.
CHAPTER 3

A REFINEMENT OF THE STURM-LIOUVILLE
CONNECTION FORMULAE

3.1 Introduction

In this chapter we consider connection formulae for the spectral functions $\rho_{\alpha}(\mu)$ for $\mu \in \mathbb{R}$ associated with the Sturm-Liouville equation

\[-(py')' + qy = \lambda wy\]  \hspace{1cm} (3.1.1)

on $[0, 1]$ together with the boundary condition

\[y(0) \cos \alpha + p(0)y'(0) \sin \alpha = 0\]  \hspace{1cm} (3.1.2)

where $\alpha \in [0, \pi)$. The spectral parameter is written $\lambda = \mu + i\epsilon$, with $\mu, \epsilon \in \mathbb{R}$. Two variations of the problem are considered.

(I) Equation (3.1.1) is in the limit-point case at infinity and the coefficient functions $p, q$ and $w$ satisfy

i. $p, q, w$ are real valued on $[0, \infty)$,

ii. $w(x) > 0$ and $p(x) > 0$ for $x \in [0, \infty)$,

iii. $1/p, q, w \in L^1_{\text{loc}}[0, \infty)$.

(II) The real valued functions $p, q$ and $w$ in (3.1.1) are such that in addition to the requirements of (I),
i. there is a \( \Lambda_0 \in \mathbb{R} \) such that for all \( \mu \geq \Lambda_0 \), the spectral derivative \( \rho'_\alpha(\mu) \) is continuous and \( 0 < \rho'_\alpha(\mu) < \infty \) for \( \alpha \in [0, \pi) \).

ii. there exist real valued functions \( S(\mu), T(\mu) \) such that for \( \mu \geq \Lambda_0 \),

\[
\rho'_\alpha(\mu) = \frac{1}{\pi} \frac{T(\mu)}{(S(\mu)^2 + T(\mu)^2) \sin^2 \alpha + S(\mu) \sin 2\alpha + \cos^2 \alpha}
\] (3.1.3)

with \( S(\mu) \to 0 \) and \( T(\mu) \to \mu^{1/2} \) as \( \mu \to \infty \).

Condition (II) is satisfied for special cases of the Sturm-Liouville equation with \( p \equiv w \equiv 1 \) and, for example, either of the following conditions on \( q \).

a. \( q \in L^1[0, \infty) \) and there exists a \( \Lambda_0 \in \mathbb{R} \) and functions \( a(x) \) and \( \eta(\mu) \) such that

\[
\left| \int_x^\infty \exp(2i\mu^{1/2}t) q(t) \, dt \right| \leq a(x) \eta(\mu) \quad \text{for} \quad \mu \geq \Lambda_0 \quad \text{and} \quad 0 \leq x < \infty
\]

where \( a(\cdot) \) is a decreasing \( L^1[0, \infty) \) function and \( \eta(\mu) \to 0 \) as \( \mu \to \infty \). See [7, 9, 11].

b. Conditions (Q) of §2.2.

Condition (I) places minimal restrictions on (3.1.1) and the results of Gilbert and Harris [7] and Eastham [4] apply. It is known that the spectral derivatives for three initial conditions \( \alpha, \alpha + \pi/2 \), and \( \beta \) are related and that given the spectral derivatives for two initial conditions, the third derivative must satisfy a quadratic equation. Here, this result is given in greater generality and a corollary is proved. The primary result of this chapter, Theorem 3.3, states that when condition (II) is satisfied, a third derivative is uniquely determined in most cases.

**Theorem 3.1** For the problem (I) and almost all \( \mu \in \mathbb{R} \), if there is an \( \alpha \in [0, \pi) \) such that \( 0 < \rho'_\alpha(\mu) < \infty \) and if \( \alpha, \beta, \) and \( \gamma \) are distinct members of \( [0, \pi) \), then
\( \rho'_\alpha(\mu), \rho'_\beta(\mu) \) exist with \( 0 < \rho'_\beta(\mu) < \infty \), \( 0 < \rho'_\gamma(\mu) < \infty \) and

\[
\left[ \frac{\sin^2(\beta - \alpha)}{\rho'_\alpha(\mu)} - \frac{\sin^2(\beta - \gamma)}{\rho'_\alpha(\mu)} - \frac{\sin^2(\gamma - \alpha)}{\rho'_\beta(\mu)} \right]^2 = 4 \sin^2(\beta - \gamma) \sin^2(\gamma - \alpha) \left( \frac{1}{\rho'_\alpha(\mu)\rho'_\beta(\mu)} - \pi^2 \sin^2(\beta - \alpha) \right). \tag{3.1.4}
\]

Thus, \( \frac{\sin^2(\beta - \alpha)}{\rho'_\gamma(\mu)} \) must be one of the two choices

\[
\frac{\sin^2(\beta - \gamma)}{\rho'_\alpha(\mu)} + \frac{\sin^2(\gamma - \alpha)}{\rho'_\beta(\mu)} \pm 2 \sin(\beta - \gamma) \sin(\gamma - \alpha) \sqrt{\frac{1}{\rho'_\alpha(\mu)\rho'_\beta(\mu)} - \pi^2 \sin^2(\beta - \alpha)}.
\tag{3.1.5}
\]

Equation (3.1.4) is a more general form of (1.2.9), the connection formula for three initial conditions given by Gilbert and Harris in [7]. A corollary is as follows.

**Corollary 3.2** For the problem (I), if \( \rho'_\alpha(\mu), \rho'_\beta(\mu) \) exist as finite positive numbers for distinct \( \alpha, \beta \in [0, \pi) \), then

\[
\rho'_\alpha(\mu)\rho'_\beta(\mu) \leq \frac{1}{\pi^2 \sin^2(\beta - \alpha)} \tag{3.1.6}
\]

For the problem (II), this result is true for all \( \mu \geq \Lambda_0 \).

The cases of the following theorem arise depending on whether the spectral derivative for the initial condition 0 is known. There are three cases: \( \alpha, \beta \) are both nonzero, one of \( \alpha, \beta \) is zero and the other is in \((0, \pi/2)\), and one is zero and the other is in \((\pi/2, \pi)\). Since \( \alpha, \beta \) play interchangeable roles, we assume \( \alpha < \beta \).

**Theorem 3.3** For the problem (II) and \( \mu \geq \Lambda_0 \),
(i) if $\alpha, \beta \in (0, \pi)$ and $\gamma \in [0, \pi)$ or if $\alpha = 0$, $\beta \in (0, \pi/2)$ and $\gamma \in (0, \pi)$ then

$$
\frac{\sin^2(\beta - \alpha)}{\rho'_\gamma(\mu)} = \frac{\sin^2(\beta - \gamma)}{\rho'_\alpha(\mu)} + \frac{\sin^2(\gamma - \alpha)}{\rho'_\beta(\mu)} + 2 \sin (\beta - \gamma) \sin (\gamma - \alpha) \sqrt{\frac{1}{\rho'_\alpha(\mu)\rho'_\beta(\mu)}} - \pi^2 \sin^2 (\beta - \alpha).
$$

(ii) if $\alpha = 0, \beta \in (\pi/2, \pi)$ and $\gamma \in (0, \pi)$ then

$$
\frac{\sin^2(\beta - \alpha)}{\rho'_\gamma(\mu)} = \frac{\sin^2(\beta - \gamma)}{\rho'_\alpha(\mu)} + \frac{\sin^2(\gamma - \alpha)}{\rho'_\beta(\mu)} - 2 \sin (\beta - \gamma) \sin (\gamma - \alpha) \sqrt{\frac{1}{\rho'_\alpha(\mu)\rho'_\beta(\mu)}} - \pi^2 \sin^2 (\beta - \alpha).
$$

We use the principal branch of the square root function. Theorem 3.3 omits the case where $\alpha = 0$ and $\beta = \pi/2$. In this situation, additional information on the sign of $S(\mu)$ is required before $\rho'_\gamma(\mu)$ can be determined uniquely, as will be made clear in the proof. The results are proved in §3.2 and examples are given in §3.3.

### 3.2 Proofs of the Connection Formulae

**Proof of Theorem 3.1.** For almost all $\mu \in \mathbb{R}$ for which $\rho'_\alpha(\mu)$ exists with $0 < \rho'_\alpha(\mu) < \infty$, $m_\alpha(\mu + i\epsilon)$ converges to a finite non-real limit as $\epsilon \to 0^+$ and satisfies (1.2.7), where $m_\alpha(\lambda)$ is the Titchmarsh-Weyl function for $\lambda$ in the upper half-plane. For such $\mu$, let $X_\alpha(\mu) = \lim_{\epsilon \to 0^+} \text{Re}\{m_\alpha(\mu + i\epsilon)\}$. Then,

$$
m_\alpha(\mu) = \lim_{\epsilon \to 0^+} m_\alpha(\mu + i\epsilon) = X_\alpha(\mu) + i\pi m'_\alpha(\mu)
$$

where the imaginary part of (3.2.1) follows from (1.2.7). Since $\lim_{\epsilon \to 0^+} m_\alpha(\mu)$ exists as a finite non-real limit if and only if the same is true of $\lim_{\epsilon \to 0^+} m_\beta(\mu)$ (see [7],
Lemma 1.1), it follows from (1.2.7) that for \( \mu \) satisfying (3.2.1), \( \rho'_\beta(\mu) \) also exists with \( 0 < \rho'_\beta(\mu) < \infty \) for all \( \beta \in [0, \pi) \). Hence by [7, Equation (5.6)] we have

\[
\frac{\rho'_\beta(\mu)}{\rho'_\alpha(\mu)} = \lim_{\epsilon \to 0^+} \frac{1}{|m_\alpha(\mu + i\epsilon) \sin(\beta - \alpha) + \cos(\beta - \alpha)|^2}
\]

\[
= \frac{1}{|X_\alpha(\mu) \sin(\beta - \alpha) + \cos(\beta - \alpha) + i\pi\rho'_\alpha(\mu) \sin(\beta - \alpha)|^2}
\]

So

\[
\frac{\rho'_\alpha(\mu)}{\rho'_\beta(\mu)} = (X_\alpha(\mu) \sin(\beta - \alpha) + \cos(\beta - \alpha))^2 + \pi^2\rho'_\alpha(\mu)^2 \sin^2(\beta - \alpha) \tag{3.2.2}
\]

and

\[
0 = X_\alpha(\mu)^2 \sin^2(\beta - \alpha) + X_\alpha(\mu) \sin(2\beta - 2\alpha)
\]

\[
+ \cos^2(\beta - \alpha) + \pi^2\rho'_\alpha(\mu)^2 \sin^2(\beta - \alpha) - \frac{\rho'_\alpha(\mu)}{\rho'_\beta(\mu)}
\]

The last equation is quadratic in \( X_\alpha(\mu) \), so \( X_\alpha(\mu) \) is one of

\[
- \cos(\beta - \alpha) \pm \frac{\sqrt{\frac{\rho'_\alpha(\mu)}{\rho'_\beta(\mu)}} - \pi^2\rho'_\alpha(\mu)^2 \sin^2(\beta - \alpha)}}{\sin(\beta - \alpha)} \tag{3.2.3}
\]

Equation (3.2.2) is also valid when \( \beta \) is replaced by \( \gamma \), hence, using (3.2.3) for \( X_\alpha(\mu) \), the two choices for \( \frac{\rho'_\alpha(\mu)}{\rho'_\gamma(\mu)} \) are

\[
\left( \sin(\gamma - \alpha) \left( - \cos(\beta - \alpha) \pm \frac{\sqrt{\frac{\rho'_\alpha(\mu)}{\rho'_\beta(\mu)}} - \pi^2\rho'_\alpha(\mu)^2 \sin^2(\beta - \alpha)}}{\sin(\beta - \alpha)} \right) + \cos(\gamma - \alpha) \right)^2
\]

\[
+ \pi^2\rho'_\alpha(\mu)^2 \sin^2(\gamma - \alpha).
\]
This is equivalent to

\[
\frac{\rho'_\alpha(\mu)}{\rho'_\beta(\mu)} = \left( \frac{\sin(\beta - \gamma) \pm \sin(\gamma - \alpha) \sqrt{\frac{\rho'_\alpha(\mu)}{\rho'_\beta(\mu)} - \pi^2 \sin^2(\beta - \alpha)}}{\sin(\beta - \alpha)} \right)^2 + \pi^2 \rho'_\alpha(\mu)^2 \sin^2(\gamma - \alpha)
\]

and the formulae (3.1.4)–(3.1.5) follow upon rearranging. Q.E.D.

**Proof of Corollary 3.2.** By Theorem 1.2, if \(0 < \rho'_\alpha(\mu) < \infty\) and \(0 < \rho'_\beta(\mu) < \infty\) for distinct \(\alpha, \beta \in [0, \pi]\), then \(0 < \rho'_\gamma(\mu) < \infty\) for all \(\gamma \in [0, \pi]\). Hence, (3.1.5) must give real numbers. That is,

\[
\frac{1}{\rho'_\alpha(\mu)\rho'_\beta(\mu)} - \pi^2 \sin^2(\beta - \alpha) \geq 0
\]

(3.2.4)

from which the result follows. Note that for the problem (II), the inequality holds for all \(\mu \geq \Lambda_0\) since the spectral derivatives are assumed finite and positive for these \(\mu\) and all \(\alpha \in [0, \pi]\). Q.E.D.

**Proof of Theorem 3.3.** In the case of (II), we can write \(\rho'_\alpha(\mu), \rho'_\beta(\mu)\) in terms of \(S(\mu), T(\mu)\) using (3.1.3). The radicand in (3.1.5) is a perfect square:

\[
\sqrt{\frac{1}{\rho'_\alpha(\mu)\rho'_\beta(\mu)} - \pi^2 \sin^2(\beta - \alpha)} = \frac{\pi}{T} \left| (S^2 + T^2) \sin \alpha \sin \beta + S \sin(\alpha + \beta) + \cos \alpha \cos \beta \right|
\]

(3.2.5)

We will see that correctly selecting the branch of the square root depends on whether the quantity within absolute value in (3.2.5) is negative or nonnegative. Temporarily,
let us suppose it is nonnegative. Then the positive branch of (3.1.5) can be written

\[
\frac{\pi}{T} \left\{ (S^2 + T^2) \sin \alpha \sin (\beta - \gamma) + \sin \beta \sin (\gamma - \alpha))^2
\right.

+ S \sin^2 (\beta - \gamma) \sin 2\alpha + \sin^2 (\gamma - \alpha) \sin 2\beta + 2 \sin (\beta - \gamma) \sin (\gamma - \alpha) \sin (\alpha + \beta)

\left. + (\cos \alpha \sin (\beta - \gamma) + \cos \beta \sin (\gamma - \alpha))^2 \right\}.
\]

Trigonometric identities are used to simplify this expression. The coefficient of \((S^2 + T^2)\) becomes

\[
\left( \sin \alpha \sin (\beta - \gamma) + \sin \beta \sin (\gamma - \alpha) \right)^2
\]

\[
= (\sin \alpha (\sin \beta \cos \gamma - \cos \beta \sin \gamma) + \sin \beta (\sin \gamma \cos \alpha - \cos \gamma \sin \alpha))^2
\]

\[
= (\sin \gamma (\sin \beta \cos \alpha - \cos \alpha \cos \beta))^2
\]

\[
= \sin^2 \gamma \sin^2 (\beta - \alpha).
\]

The constant term is simplified in a similar way and the coefficient of \(S\) becomes

\[
\sin^2 (\beta - \gamma) \sin 2\alpha + \sin^2 (\gamma - \alpha) \sin 2\beta + 2 \sin (\beta - \gamma) \sin (\gamma - \alpha) \sin (\alpha + \beta)
\]

\[
= \frac{1}{2} \sin (\beta - \gamma) \left[ \cos (2\alpha - \beta + \gamma) - \cos (\gamma + \beta) \right]
\]

\[
- \frac{1}{2} \sin (\gamma - \alpha) \left[ \cos (2\beta + \gamma - \alpha) - \cos (\gamma + \alpha) \right]
\]

\[
= \sin (\beta - \gamma) \sin (\alpha + \gamma) \sin (\beta - \alpha) + \sin (\gamma - \alpha) \sin (\beta + \gamma) \sin (\beta - \alpha)
\]

\[
= \sin (\beta - \alpha) \left[ \frac{1}{2} \cos (\beta - 2\gamma - \alpha) - \frac{1}{2} \cos (\beta + 2\gamma - \alpha) \right]
\]

\[
= \sin^2 (\beta - \alpha) \sin 2\gamma
\]
That is, the positive branch of (3.1.5) simplifies to

\[ \frac{\pi}{T} \left\{ (S^2 + T^2) \sin^2(\beta - \alpha) \sin^2 \gamma + S \sin^2(\beta - \alpha) \sin 2\gamma + \sin^2(\beta - \alpha) \cos^2 \gamma \right\} \]

which is identically \( \frac{\sin^2(\beta - \alpha)}{\rho'_{\gamma}(\mu)} \) by (3.1.3). Therefore, when the quantity within absolute value in (3.2.5) is nonnegative, the positive branch yields the correct value for the spectral derivative. In a similar way, if the expression within absolute value is negative, then the negative branch of (3.1.5) will simplify as above and is the correct choice.

We now use the fact that \( \rho'_{\gamma}(\mu) \) is continuous for \( \mu \geq \Lambda_0 \). This implies that the same branch of the square root must be used for each \( \mu \geq \Lambda_0 \), once \( \alpha, \beta \) are fixed. That is, the sign of the quantity within absolute value in (3.2.5) can not change for \( \mu \geq \Lambda_0 \). It is clear that for \( \alpha, \beta \neq 0 \) and large \( \mu \), this quantity is positive since \( S(\mu) \to 0 \) and \( T(\mu) \to \infty \). Thus the positive branch gives the correct value. On the other hand, if \( \alpha = 0 \), the quantity within absolute value tends to \( \cos \beta \) and so the positive or negative branch of (3.1.5) is chosen depending on the quadrant in which \( \beta \) lies. This establishes formulae (3.1.7) and (3.1.8). Note if \( \alpha = 0 \), and \( \beta = \pi/2 \), then the expression in (3.2.5) is \( \frac{\pi |S|}{T} \). This tends to 0, but it is not clear whether the approach is from above or below. Hence information on the sign of \( S \) is required before the branch can be selected. Q.E.D.

### 3.3 Examples

For \( p \equiv w \equiv 1 \) and \( q = 0 \) in (3.1.1), \( \rho'_{\alpha}(\mu) \) may be computed directly and

\[ \rho'_{\alpha}(\mu) = \frac{1}{\pi} \frac{\sqrt{\mu}}{\mu \sin^2 \alpha + \cos^2 \alpha}, \quad \mu > 0. \]  

(3.3.1)
Here \( S(\mu) \equiv 0 \) and \( T(\mu) = \sqrt{\mu} \). To illustrate Theorem 3.3, we set \( \alpha = \pi/3 \) and \( \beta = \pi/2 \). Then \( \rho_0'(\mu) = \frac{4\sqrt{\mu}}{\pi 3\mu + 1} \) and \( \rho_\beta'(\mu) = \frac{1}{\pi \sqrt{\mu}} \). Since \( \alpha \) and \( \beta \) are both nonzero, the positive branch of the square root function must be used. For \( \gamma = 0 \), (3.1.7) gives

\[
\frac{1/4}{\rho_0'(\mu)} = \pi \left[ \frac{6\mu + 1}{4\sqrt{\mu}} + (2)(1) \left( \frac{-\sqrt{3}}{2} \right) \frac{\sqrt{3\mu}}{2} \right] = \frac{\pi}{4\sqrt{\mu}}.
\]

Hence, \( \rho_0'(\mu) = \frac{\sqrt{\mu}}{\pi} \), which agrees with (3.3.1). The value obtained from (3.1.8), using the negative branch of the square root function, would indicate (incorrectly) that \( \rho_0'(\mu) \) is \( \frac{1}{\pi} \frac{\sqrt{\mu}}{12\mu + 1} \). This rejected value does not have the correct asymptotic behavior as \( \mu \to \infty \).

If \( \gamma = 5\pi/6 \), then (3.1.7) gives

\[
\frac{1/4}{\rho_{5\pi/6}'(\mu)} = \pi \left[ \frac{25\mu/4 + 3/4}{4\sqrt{\mu}} + (2) \left( \frac{-\sqrt{3}}{2} \right) (1) \frac{\sqrt{3\mu}}{2} \right] = \frac{\pi \sqrt{\mu/4 + 3/4}}{4\sqrt{\mu}}.
\]

Hence, \( \rho_{5\pi/6}'(\mu) = \frac{1}{\pi} \frac{4\sqrt{\mu}}{\mu + 3} \). The rejected value is \( \frac{1}{\pi} \frac{4\sqrt{\mu}}{49\mu + 3} \) and although it has the same asymptotic behavior as the correct value, it can not be written in the form of (3.1.3). Thus to uniquely determine a spectral derivative for a third initial condition, it is not sufficient to know only the asymptotic behavior of \( \rho_\gamma'(\mu) \)—rather, knowledge of the asymptotic behavior of the component functions \( S \) and \( T \) in (3.1.3) is required.

To illustrate Theorem 3.3(ii), let \( \alpha = 0 \), \( \beta = 5\pi/6 \), and \( \gamma = \pi/3 \). Then (3.1.8)
gives
\[
\frac{1/4}{\rho'_{\pi/3}(\mu)} = \pi \left[ \frac{3\mu/4 + 25/4}{4\sqrt{\mu}} - (2)(1) \left( \frac{\sqrt{3}}{2} \right) \sqrt{\frac{3}{4\mu}} \right]
\]
\[
= \pi \frac{3\mu/4 + 1/4}{4\sqrt{\mu}}.
\]

Hence, \( \rho'_{\pi/3}(\mu) = \frac{1}{\pi} \frac{4\sqrt{\mu}}{3\mu + 1} \).

Finally, consider the inequality in Corollary 3.2. By (3.1.3), \( \rho'_{\pi/2}(\mu) = \frac{T(\mu)}{\pi(S(\mu)^2 + T(\mu)^2)} \) and \( \rho'_0(\mu) = \frac{T(\mu)}{\pi} \). Hence, for \( \alpha = 0, \beta = \pi/2 \), the bound of (3.1.6) is attained when \( S(\mu) = 0 \). This occurs, for example, when \( p \equiv w \equiv 1, q = 0 \). Note that in the case of (II) for \( \alpha, \beta \neq 0 \), the left side of (3.1.6) tends to 0 as \( \mu \to \infty \) while the right side is at least \( 1/\pi^2 \). The inequality is more significant when \( \alpha = 0 \) for then the limit of the left side is not 0.
CHAPTER 4

CONNECTION FORMULAE FOR SPECTRAL FUNCTIONS ASSOCIATED WITH SINGULAR DIRAC EQUATIONS

4.1 Introduction

In this chapter, we consider connection formulae for spectral functions $\rho_\mu'(\mu), \mu \in \mathbb{R}$, associated with the Dirac equation given by

$$y' = \begin{pmatrix} p & \lambda + c + v_1 \\ - (\lambda - c + v_2) & -p \end{pmatrix} y, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad (4.1.1)$$

on $[0, \infty)$ together with the initial condition

$$y_1(0) \cos \alpha + y_2(0) \sin \alpha = 0 \quad (4.1.2)$$

where $\alpha \in [0, \pi)$. Also, $c \geq 0$ is a constant, $\lambda = \mu + i\epsilon$ is the complex spectral parameter, and $v_1, v_2,$ and $p$ are real valued members of $L^1[0, \infty)$. To each parameter $\alpha \in [0, \infty)$, one may define $\theta_\alpha$ and $\varphi_\alpha$ as solutions of (4.1.1) that satisfy for all $\lambda$

$$\theta_\alpha(0, \lambda) = \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}, \quad \varphi_\alpha(0, \lambda) = \begin{pmatrix} - \sin \alpha \\ \cos \alpha \end{pmatrix}. \quad (4.1.3)$$

Then the Titchmarsh-Weyl $m_\alpha(\lambda)$ function is defined by

$$\psi_\alpha(x, \lambda) = \theta_\alpha(x, \lambda) + m_\alpha(\lambda)\varphi_\alpha(x, \lambda) \in L^2[0, \infty). \quad (4.1.4)$$
We assume (4.1.1) is in the limit point case at infinity, which implies that $m_\alpha(\lambda)$ is well defined and unique for $\text{Im}\{\lambda\} > 0$. Also, in the limit point case, the $L^2(0, \infty)$ solution of (4.1.1) is unique up to constant multiples; thus it follows that $\psi_\alpha(x, \lambda)$ and $\psi_\beta(x, \lambda)$ are linearly dependent. Following Hille [12], the Wronskian of $\psi_\alpha(0, \lambda)$ and $\psi_\beta(0, \lambda)$ is 0, that is,

$$\begin{vmatrix} \cos \alpha - m_\alpha(\lambda) \sin \alpha & \cos \beta - m_\beta(\lambda) \sin \beta \\ \sin \alpha + m_\alpha(\lambda) \cos \alpha & \sin \beta + m_\beta(\lambda) \cos \beta \end{vmatrix} = 0. \quad (4.1.5)$$

This gives the $m$ connection formula

$$m_\beta(\lambda) = \frac{m_\alpha(\lambda) \cos(\beta - \alpha) - \sin(\beta - \alpha)}{m_\alpha(\lambda) \sin(\beta - \alpha) + \cos(\beta - \alpha)}. \quad (4.1.6)$$

The spectral functions may then be defined in terms of these $m_\alpha(\lambda)$ functions through the Titchmarsh-Kodaira formula (1.2.7).

The purpose of this chapter is to show how the spectral derivatives $\rho'_\alpha(\mu)$ of (4.1.1) for distinct initial conditions are related. The assumptions are minimal: the equation must be in the limit point case at infinity and $\mu$ must be such that $0 < \rho'_\alpha(\mu) < \infty$ for all $\alpha \in [0, \infty)$. Hinton and Shaw [14] prove these assumptions are met when, for example, $p, v_1$, and $v_2$ are integrable and $|\mu| > c$.

It is notable that the connection formulae for (4.1.1) are superficially similar to those for the Sturm-Liouville equation (3.1.1) considered in Chapter 3 and by Gilbert and Harris in [7]. It will be seen in Chapter 5 that under additional assumptions on $p, v_1$, and $v_2$, the spectral derivatives of (4.1.1) all tend to $\frac{1}{\pi}$ as $|\mu| \to \infty$. Recall that in the Sturm-Liouville case, with appropriate restrictions on $p, q$, and $w$, $\rho'_\alpha(\mu) \sim \pi^{-1}\mu^{\pm 1/2}$, depending on whether $\alpha = 0$ or $\alpha \in (0, \pi)$. Despite the $\rho'_\alpha(\mu)$ having distinctly different behaviors in the two cases, their connection formulae have the same structure.
The results are given in §4.2 and proved in §4.3. Examples are given in §4.4.

4.2 The Connection Formulae

**Theorem 4.1** For almost all \( \mu \), if there is an \( \alpha \in [0, \pi) \) such that \( \rho'_\alpha(\mu) \) exists with 
\[ 0 < \rho'_\alpha(\mu) < \infty, \]
then

(i) \( \rho'_\beta(\mu) \) exists with \( 0 < \rho'_\beta(\mu) < \infty \) for all \( \beta \in [0, \pi) \).

(ii) For distinct \( \alpha, \beta, \gamma, \delta \in [0, \pi) \), the spectral derivatives associated with (4.1.1) satisfy

\[
0 = \frac{\sin(\beta - \gamma) \sin(\gamma - \delta) \sin(\delta - \beta)}{\rho'_\alpha(\mu)} - \frac{\sin(\gamma - \delta) \sin(\delta - \alpha) \sin(\alpha - \gamma)}{\rho'_\beta(\mu)} + \frac{\sin(\delta - \alpha) \sin(\alpha - \beta) \sin(\beta - \delta)}{\rho'_\gamma(\mu)} - \frac{\sin(\alpha - \beta) \sin(\beta - \gamma) \sin(\gamma - \alpha)}{\rho'_\delta(\mu)}. \tag{4.2.1}
\]

As in the Sturm-Liouville equation case, if \( \rho'_\alpha(\mu), \rho'_\beta(\mu) \) exist and satisfy \( 0 < \rho'_\alpha(\mu) < \infty, 0 < \rho'_\beta(\mu) < \infty \) for distinct \( \alpha, \beta \in [0, \pi) \), then \( \rho'_\gamma(\mu) \) exists and satisfies \( 0 < \rho'_\gamma(\mu) < \infty \), for all \( \gamma \in [0, \pi) \). See [7, Remark 5.2].

**Theorem 4.2** If there are distinct \( \alpha, \beta \in [0, \pi) \) such that \( \rho'_\alpha(\mu), \rho'_\beta(\mu) \) exist with 
\[ 0 < \rho'_\alpha(\mu) < \infty, 0 < \rho'_\beta(\mu) < \infty, \]
then, for any \( \gamma \in [0, \pi) \), the spectral derivatives associated with (4.1.1) satisfy

\[
\left[ \frac{\sin^2(\beta - \alpha)}{\rho'_\gamma(\mu)} - \frac{\sin^2(\beta - \gamma)}{\rho'_\alpha(\mu)} - \frac{\sin^2(\gamma - \alpha)}{\rho'_\beta(\mu)} \right]^2 = 4 \sin^2(\beta - \gamma) \sin^2(\gamma - \alpha) \left( \frac{1}{\rho'_\alpha(\mu) \rho'_\beta(\mu)} - \pi^2 \sin^2(\beta - \alpha) \right). \tag{4.2.2}
\]
Thus, \[
\frac{\sin^2(\beta - \alpha)}{\rho'_\alpha(\mu)}
\] must be one of the two choices

\[
\frac{\sin^2(\beta - \gamma)}{\rho'_\alpha(\mu)} + \frac{\sin^2(\gamma - \alpha)}{\rho'_\beta(\mu)} + 2\sin(\beta - \gamma)\sin(\gamma - \alpha)\sqrt{\frac{1}{\rho'_\alpha(\mu)\rho'_\beta(\mu)}} - \pi^2\sin^2(\beta - \alpha).
\] (4.2.3)

**Corollary 4.3** If \(0 < \rho'_\alpha(\mu) < \infty\), \(0 < \rho'_\beta(\mu) < \infty\) for distinct \(\alpha, \beta \in [0, \pi]\), then

\[
\rho'_\alpha(\mu)\rho'_\beta(\mu) \leq \frac{1}{\pi^2\sin^2(\beta - \alpha)}.
\] (4.2.4)

Theorem 4.1 is the analog of Theorem 2.1 in [7] in the form given by Eastham [4]. Similarly, Theorem 4.2 is comparable to Theorem 2.2 in [7] where there need be no special relationship among the three initial conditions. The analog to the corollary is Corollary 3.2 of Chapter 3.

In [4], Eastham obtains several relationships among the spectral derivatives associated with Sturm-Liouville equation for special values of \(\alpha, \beta, \gamma, \delta\). Analogous corollaries are valid for the Dirac equation (4.1.1) and are listed here.

**Corollary 4.4** Suppose \(\rho'_\alpha(\mu)\) exists and satisfies \(0 < \rho'_\alpha(\mu) < \infty\) for all \(\alpha \in [0, \pi]\). Then

\[
\left(\frac{1}{\rho'_\alpha(\mu)} - \frac{1}{\rho'_{\alpha+\pi/2}(\mu)}\right)
\] does not depend on \(\alpha\).

**Corollary 4.5** Suppose \(\rho'_\alpha(\mu)\) exists and satisfies \(0 < \rho'_\alpha(\mu) < \infty\) for all \(\alpha \in [0, \pi]\). Then for any fixed \(\eta\),

\[
\left(\frac{1}{\rho'_{\eta+\alpha}(\mu)} - \frac{1}{\rho'_{\eta-\alpha}(\mu)}\right)\csc 2\alpha
\] does not depend on \(\alpha, (\alpha \neq 0, \pi/2)\).
We adopt the convention of using mod π values if the parameter falls outside the interval [0, π). As in [4], the proofs of these corollaries follow quickly from (4.2.1) with \( \alpha + \beta = \gamma + \delta \).

Spectral functions are related to the Titchmarsh-Weyl \( m_\alpha(\lambda) \) functions and, for different initial conditions, these \( m \) functions are connected by (1.2.4). Further, for special cases of the Sturm-Liouville and Dirac equations, it is possible to relate the spectral functions to solutions of a Riccati equation. (See [11] and Chapter 2 for the Sturm-Liouville case and Chapter 5 for the Dirac case.) Since it is known that the cross ratio of four solutions of a Riccati equation is constant, [15], it is reasonable to expect that the spectral functions are also connected. The results of [7], Chapter 3, and the present chapter bear out that prediction and also show the structure of the connection formulae for the two equations is similar. The degree of similarity is surprising given that the Riccati equations and the spectral functions for the two equations are very different. Additional insight might be gained by examining these connection formulae from a geometric view.

4.3 Proofs

**Proof of Theorem 4.1.** (i) The proof of this is the same as the proof of the corresponding statement in the Sturm-Liouville equation case; see [7, Theorem 2.1i]. The properties of Herglotz functions and (1.2.7) imply that for almost all \( \mu \in \mathbb{R} \), if \( \rho'_\alpha(\mu) \) exists with \( 0 < \rho'_\alpha(\mu) < \infty \), then \( m_\alpha(\mu + i\epsilon) \) converges to a finite nonreal limit as \( \epsilon \to 0^+ \) in which case by (4.1.6), \( m_\beta(\mu + i\epsilon) \) also converges to a finite nonreal limit as \( \epsilon \to 0^+ \). It then follows from (1.2.7) that \( \rho'_\beta(\mu) \) exists and has the required properties, for all \( \beta \in [0, \pi) \).

(ii) \( m_\alpha(\mu + i\epsilon) \) converges to a finite non-real limit as \( \epsilon \to 0^+ \) for almost all \( \mu \) for
which \(\rho'_\alpha(\mu)\) exists with \(0 < \rho'_\alpha(\mu) < \infty\) for some \(\alpha \in [0, \pi]\). For such a \(\mu\) we denote

\[
m_\alpha(\mu) = \lim_{\epsilon \to 0^+} m_\alpha(\mu + i\epsilon) = X_\alpha(\mu) + i\pi\rho'_\alpha(\mu)
\]

(4.3.1)

where \(X_\alpha\) is real valued and the imaginary part follows from the Titchmarsh-Kodaira formula. Then by (1.2.7) and (4.1.6)

\[
\pi\rho'_\beta(\mu) = Im \left\{ \frac{(X_\alpha(\mu) + i\pi\rho'_\alpha(\mu)) \cos(\beta - \alpha) - \sin(\beta - \alpha)}{(X_\alpha(\mu) + i\pi\rho'_\alpha(\mu)) \sin(\beta - \alpha) + \cos(\beta - \alpha)} \right\}
\]

\[
= \frac{\pi\rho'_\alpha(\mu)}{|X_\alpha(\mu) \sin(\beta - \alpha) + \cos(\beta - \alpha) + i\pi\rho'_\alpha(\mu) \sin(\beta - \alpha)|^2}
\]

(4.3.2)

Hence

\[
\frac{\rho'_\alpha(\mu)}{\rho'_\beta(\mu)} - 1 = \left( X_\alpha(\mu)^2 + \pi^2 \rho'_\alpha(\mu)^2 - 1 \right) \sin^2(\beta - \alpha) + X_\alpha(\mu) \sin(2\beta - \alpha).
\]

(4.3.3)

The coefficients of \(\sin^2(\beta - \alpha)\) and \(\sin 2(\beta - \alpha)\) depend on \(\alpha\) and \(\mu\), not \(\beta\). So replacing \(\beta\) in turn by \(\gamma\) and \(\delta\) in (4.3.3) gives three equations in two variables which must therefore be linearly dependent. Thus,

\[
\det \begin{pmatrix}
\frac{\rho'_\alpha(\mu)}{\rho'_\beta(\mu)} - 1 & \sin^2(\beta - \alpha) & \sin(2\beta - \alpha) \\
\frac{\rho'_\alpha(\mu)}{\rho'_\gamma(\mu)} - 1 & \sin^2(\gamma - \alpha) & \sin(2\gamma - \alpha) \\
\frac{\rho'_\alpha(\mu)}{\rho'_\delta(\mu)} - 1 & \sin^2(\delta - \alpha) & \sin(2\delta - \alpha)
\end{pmatrix} = 0.
\]

(4.3.4)

Formula (4.2.1) follows upon expanding about the first column and using trigonometric identities. Specifically, note that

\[
\sin^2(c - a) \sin 2(d - a) - \sin^2(d - a) \sin 2(c - a)
\]
\[ = -\frac{1}{2} (\sin(2c - 2a) + \sin(2a - 2d) + \sin(2d - 2c)) \]
\[ = 2 \sin(c - a) \sin(d - a) \sin(c - d) \]

and that

\[-2 \sin(c - a) \sin(d - a) \sin(c - d) + 2 \sin(b - a) \sin(d - a) \sin(b - d) \]
\[-2 \sin(c - d) \sin(b - d) \sin(c - b) = 2 \sin(c - d) \sin(b - d) \sin(c - b) \]

This completes the proof of Theorem 4.1. \[ \text{Q.E.D.} \]

**Proof of Theorem 4.2.** As in the proof of Theorem 4.1, \( \rho'_\alpha(\mu) \), \( \rho'_\beta(\mu) \) and \( X_\alpha(\mu) = \text{Re}\{m_\alpha(\mu)\} \) are related by equation (4.3.2). Rearranging yields an equation quadratic in \( X_\alpha \):

\[ 0 = X_\alpha(\mu)^2 \sin^2(\beta - \alpha) + \pi^2 \rho'_\alpha(\mu)^2 \sin^2(\beta - \alpha) - \frac{\rho'_\alpha(\mu)}{\rho'_\beta(\mu)} \]

So \( X_\alpha(\mu) \) is one of

\[ -\cos(\beta - \alpha) \pm \frac{\sqrt{\rho'_\beta(\mu)} + \pi^2 \rho'_\alpha(\mu)^2 \sin^2(\beta - \alpha)}}{\sin(\beta - \alpha)}. \]  \( (4.3.5) \)

Equation (4.3.2) remains valid when \( \beta \) is replaced by \( \gamma \). Replacing \( X_\alpha(\mu) \) by (4.3.5) in this expression and rearranging gives formulae (4.2.2) and (4.2.3). \[ \text{Q.E.D.} \]

**Proof of Corollary 4.3.** If \( 0 < \rho'_\alpha(\mu) < \infty \), \( 0 < \rho'_\beta(\mu) < \infty \), then by Theorem 4.2, \( \rho'_\gamma(\mu) \) exists for all \( \gamma \in [0, \pi) \). So the radicand in (4.2.3) must be nonnegative and the result follows. \[ \text{Q.E.D.} \]
4.4 Examples

If the coefficient functions \( p, v_1, \) and \( v_2 \in L^1[0, \infty) \), then \( 0 < \rho'_{\alpha}(\mu) < \infty \) for all \( |\mu| > c \) and \( \alpha \in [0, \pi) \). See [13, 14]. Hence the connection formulae of this chapter hold. As a particular example, we take \( p \equiv v_1 \equiv v_2 \equiv 0 \). Then the spectral derivatives can be computed directly from (1.2.7) and (4.1.4) and

\[
\rho'_{\alpha}(\mu) = \begin{cases} 
\frac{\sqrt{\mu^2 - c^2}}{\pi(\mu + c \cos 2\alpha)} & \mu > c \\
\frac{-\sqrt{\mu^2 - c^2}}{\pi(\mu + c \cos 2\alpha)} & \mu < -c
\end{cases}
\] (4.4.1)

for \( \alpha \in [0, \pi) \).

To illustrate Theorem 4.2, we take \( \alpha = \pi/4, \beta = \pi/2 \) and compute the choices for \( \rho_0'(\mu) \). Since, for \( \mu > c \)

\[
\rho'_{\pi/4}(\mu) = \frac{1}{\pi} \frac{\sqrt{\mu^2 - c^2}}{\mu} \quad \text{and} \quad \rho'_{\pi/2}(\mu) = \frac{1}{\pi} \frac{\sqrt{\mu^2 - c^2}}{\mu - c},
\]

Theorem 4.2 yields

\[
\left[ \frac{1}{2 \rho_0'(\mu)} - \frac{\pi}{2} \frac{3\mu - c}{\sqrt{\mu^2 - c^2}} \right]^2 = \pi^2 \left( \frac{\mu - c}{\mu + c} \right)
\]

and the choices for \( \rho_0'(\mu), \mu > c \) are

\[
\frac{1}{\pi} \frac{\sqrt{\mu^2 - c^2}}{5\mu - 3c} \quad \text{and} \quad \frac{1}{\pi} \frac{\sqrt{\mu^2 - c^2}}{\mu + c}.
\]

If a third derivative is known, say \( \rho'_{\pi/3}(\mu) = \frac{\sqrt{\mu^2 - c^2}}{\mu - c/2} \), the ambiguity is resolved and \( \rho_0'(\mu) = \frac{1}{\pi} \frac{\sqrt{\mu^2 - c^2}}{\mu + c} \) for \( \mu > c \) by Theorem 4.1. The calculation for \( \mu < -c \) is similar.
It will be seen in Chapter 6 that it is possible in some cases to uniquely determine a third derivative given derivatives for just two initial conditions.

Finally, in this simple example with \( p \equiv v_1 \equiv v_2 \equiv 0 \), equality in Corollary 4.3, is attained when \( \alpha = 0, \beta = \pi/2 \). In Chapter 5, \( \rho'_\alpha(\mu) \) for a special case of the Dirac equation will be expressed in terms of two functions \( S(\mu) \) and \( T(\mu) \) as was done by Harris for a special case of the Sturm-Liouville equation, [11]. In this notation, it is clear that whenever \( S(\mu) = 0 \) and \( \alpha = 0, \beta = \pi/2, (3.1.6) \) is an equality.
CHAPTER 5

THE FORM OF THE SPECTRAL FUNCTIONS
ASSOCIATED WITH DIRAC EQUATIONS

5.1 Introduction

In this chapter, we consider the spectral functions $\rho_\alpha(\mu)$ for $\mu \in \mathbb{R}$ associated with a Dirac equation given by

$$
y' = \begin{pmatrix} p & \lambda + c + v_1 \\ -(\lambda - c + v_2) & -p \end{pmatrix} y, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad (5.1.1)$$

on $[0, \infty)$ together with the initial condition

$$y_1(0) \cos \alpha + y_2(0) \sin \alpha = 0 \quad (5.1.2)$$

where $\alpha \in [0, \pi)$. Also, $c \geq 0$ is a constant, $\lambda = \mu + i\epsilon$ is the complex spectral parameter, and the real valued $p$, $v_1$ and $v_2$ are bounded and members of $L^1[0, \infty)$. This guarantees the equation is in the limit point case at infinity [13, 14]. We show that with suitable additional restrictions on the coefficient functions $p$, $v_1$, and $v_2$ and for $|\mu|$ large enough, the spectral derivative $\rho'_\alpha(\mu)$ can be written as a series. The form of this representation is similar to that of the spectral derivative of the Sturm-Liouville equation (3.1.1), yet there are striking differences in the behavior of the functions for the two equations.

The main result is given in §5.2 and the groundwork is developed in §5.3. The theorem is proved in §5.4, specific conditions for which it holds are established in
§5.5 and examples are given in §5.6. Throughout the chapter, we take $\lambda$ in the upper half-plane, $\epsilon = Im\{\lambda\} \geq 0$.

5.2 The Representation of $\rho'_\alpha(\mu)$

We make the following definitions.

\[ w = \frac{\sqrt{\lambda - c}}{\lambda + c} \quad (5.2.1) \]

\[ u_1(x, \lambda) = -\int_x^\infty e^{2i\sqrt{\lambda^2 - c^2}(t-x)} \left( w^2 v_1(t) - v_2(t) - 2iw(p(t)) \right) \, dt \quad (5.2.2) \]

\[ u_2(x, \lambda) = \int_x^\infty e^{2i\sqrt{\lambda^2 - c^2}(t-x)} \left( 2(p(t) + iwv_1(t))u_1(t, \lambda) - (\lambda + c + v_1(t))u_1(t, \lambda)^2 \right) \, dt \quad (5.2.3) \]

\[ u_n(x, \lambda) = \int_x^\infty e^{2i\sqrt{\lambda^2 - c^2}(t-x)} \left\{ 2(p(t) + iwv_1(t))u_{n-1}(t, \lambda) - (\lambda + c + v_1(t)) \left( u_{n-1}(t, \lambda)^2 + 2u_{n-1}(t, \lambda) \sum_{m=1}^{n-2} u_m(t, \lambda) \right) \right\} \, dt, \quad n \geq 3 \quad (5.2.4) \]

\[ u(x, \lambda) = iw + \sum_{n=1}^\infty u_n(x, \lambda) \quad (5.2.5) \]

\[ S(x, \lambda) = Re\{u(x, \lambda)\} \]

\[ T(x, \lambda) = Im\{u(x, \lambda)\} \]

We take the coefficient functions $p$, $v_1$, and $v_2$ to satisfy the following hypotheses.

**Hypothesis 1.** There exists a decreasing $L^1[0, \infty)$ function $a(x)$ such that

\[ |u_1(x, \lambda)| \leq \frac{a(x)}{\sqrt{\lambda^2 - c^2}}, \quad 0 \leq x < \infty \quad (5.2.6) \]

**Hypothesis 2.** Positive constants $k_0$ and $k_1$ exist satisfying each of the following.
Hypothesis 3. There exists a $\Lambda_0 > c$ such that for all $\lambda = \mu + i\epsilon$ with $\epsilon \geq 0$, $|\mu| \geq \Lambda_0$, both of the following are true.

(i) \[ \frac{4a(x)}{\lambda - c} \left( 1 + 2 \int_0^\infty |wv_1(t)| \, dt \right) < \frac{1}{k_2}, \text{ where } 0 < \frac{1}{k_2} \leq 1 - \frac{20}{k_0} - \frac{2}{k_1} \]

(ii) \[ \frac{2}{\sqrt{\lambda^2 - c^2}} \sup_{x \in [0,\infty)} \left( |p(x)| + |wv_1(x)| + \frac{2a(x)}{\sqrt{\lambda^2 - c^2}} (|\lambda + c| + |v_1(x)|) \right) \leq 1 \]

In §5.5, $a(x)$ is computed in the case that the coefficient functions and their derivatives are each monotonic $L^1[0,\infty)$ functions and in the case that the coefficient functions are each decreasing $L^1[0,\infty)$ functions. Hypothesis 2 is fairly strong to obtain absolute uniform convergence of (5.2.5). In the Dirac case, it is necessary that $|\sum u_n(x,\lambda)|$ is small relative to $|iw|$, the first term of the solution (5.2.5). The size of this term as $\lambda \to \infty$ is $|iw| = \left| i \sqrt{\frac{\lambda - c}{\lambda + c}} \right| \approx 1$. In comparison, in the Sturm-Liouville case with $p \equiv w \equiv 1$, the first term of the solution of the Riccati equation is $i\lambda^{1/2}$, which grows unboundedly. Hence the necessary conditions on $q$ are far less stringent. In Hypothesis 3, $\Lambda_0 > c$ is best possible since Hinton and Shaw prove in [14] that outside $(-c, c)$ the spectrum of (5.1.1) is continuously differentiable. Thus the limit in the Titchmarsh-Kodaira formula (1.2.7) exists as a finite and positive number.

**Theorem 5.1** If the hypotheses 1-3 are satisfied, then for $|\mu| \geq \Lambda_0$,

\[ \rho'_0(\mu) = \frac{T(0, \mu)}{\pi} \quad (5.2.7) \]
and for $\alpha \in (0, \pi)$,

$$
\rho_\alpha'(\mu) = \frac{1}{\pi} \frac{T(0, \mu)}{(S(0, \mu)^2 + T(0, \mu)^2) \sin^2 \alpha + S \sin 2\alpha + \cos^2 \alpha} 
$$

(5.2.8)

In both the Sturm-Liouville and Dirac equation cases, $S(x, \lambda) \to 0$ and $iT(x, \lambda)$ tends to the first term of the solution of the Riccati equation as $|\lambda| \to \infty$. In the Sturm-Liouville case, this means $T(x, \lambda) \to \infty$; in the Dirac case, $T(x, \lambda) \to 1$. Although the structure of the representation for $\rho_\alpha'(\mu)$ is similar in the two cases, the behavior of the functions themselves is vastly different.

Another representation for $u_n(x, \lambda)$, $n \geq 2$, obtained by a rearrangement of (5.2.3) and (5.2.4), is useful for computations and will be proved in §5.4. Suppressing arguments,

$$
u_2 = \frac{u_1^2}{2i\nu} + \int_x^\infty e^{2i\sqrt{x^2-\nu^2}(t-x)} \left\{ u_1 \left( ivv_1 - \frac{v_2}{iv} \right) - (v_1 + 2(\lambda + \nu))u_1^2 \right\} dt \tag{5.2.9}
$$

$$
\begin{align*}
\nu_n &= \frac{u_{n-1} \left( u_{n-1} + 2 \sum_{m=1}^{n-2} u_m \right)}{2i\nu} + \int_x^\infty e^{2i\sqrt{x^2-\nu^2}(t-x)} \left\{ u_{n-1} \left( ivv_1 \right. \\
&- \left. \left( u_{n-1} + 2 \sum_{m=1}^{n-2} u_m \right) v_1 + \left( \frac{1}{i\nu} \sum_{m=2}^{n-1} u'_m \right) - \frac{v_2}{i\nu} - 2(\lambda + \nu)u_1 \right) \right. \\
&\quad + \left. \frac{1}{i\nu} u'_{n-1} \sum_{m=1}^{n-2} u_m \right\} dt, \quad n \geq 3 
\end{align*}
\tag{5.2.10}
$$

5.3 The Riccati Equation

Let $\varphi_\alpha(\cdot, \lambda)$ and $\theta_\alpha(\cdot, \lambda)$ denote solutions of (5.1.1) which satisfy for all $\lambda$

$$
\begin{align*}
\theta_\alpha(0, \lambda) &= \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \\
\varphi_\alpha(0, \lambda) &= \begin{pmatrix} -\sin \alpha \\ \cos \alpha \end{pmatrix}
\end{align*}\tag{5.3.1}
$$
where \( \alpha \in [0, \pi) \). The solution \( \varphi_\alpha(x, \lambda) \) satisfies (5.1.2) and is known in the literature as the regular solution of (5.1.1). As the Dirac equation (5.1.1) is limit point, for \( \text{Im}\{\lambda\} > 0 \), there is a unique Titchmarsh-Weyl \( m_\alpha(\lambda) \) function and

\[
\theta_\alpha(x, \lambda) + m_\alpha(\lambda) \varphi_\alpha(x, \lambda) =: \psi_\alpha(x, \lambda) \in L^2[0, \infty)
\]

Suppose \( y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \) is a solution of (5.1.1) and set \( u(x, \lambda) = y_2(x, \lambda)/y_1(x, \lambda) \). Then \( u \) satisfies the Riccati equation

\[
u' = -\left(\lambda - c + v_2(x)\right) - 2p(x)u(x, \lambda) - \left(\lambda + c + v_1(x)\right)u(x, \lambda)^2. \tag{5.3.2} \]

Our aim in this section is to show that \( u \) as defined in (5.2.5) satisfies this Riccati equation. In the proof of Theorem 5.1, we will show that \( u \) is \( \psi_2/\psi_1 \), the ratio of the components of the \( L^2[0, \infty) \) solution.

As motivation for choosing the form of (5.2.5), we consider the Jost solution \( f(x, \lambda) \) of the Dirac equation (5.1.1). This solution, for \( \lambda \in \{\text{Im}\{\lambda\} \geq 0, |\lambda \pm c| \geq \epsilon, \epsilon > 0\} \), has the asymptotic behavior [13, 19]

\[
f(x, \lambda) \sim e^{i\sqrt{\lambda^2 - c^2}x} \begin{pmatrix} \frac{1}{iw} \\ 1 \end{pmatrix}
\]

so the ratio \( f_2/f_1 \to iw \) as \( x \to \infty \). Alternatively, following [13], we may consider the system consisting of the constant coefficients part of (5.1.1)

\[
\eta' = \begin{pmatrix} 0 & \lambda + c \\ -(\lambda - c) & 0 \end{pmatrix} \eta. \tag{5.3.3} \]

In this case, \( u(x, \lambda) = iw \) is easily seen to be a solution of the Riccati equation (5.3.2) and the corresponding solution of (5.3.3) satisfies \( \text{Re}\{\eta_j'/\eta_j\} = \text{Re}\{i\sqrt{\lambda^2 - c^2}\} \), \( j = \)
Lemma 5.2, below, verifies that for \( \epsilon = Im\{\lambda\} \geq 0, \ Re\{i\sqrt{\lambda^2 - c^2}\} \geq 0, \) giving (cf. argument in proof of Theorem 5.1) \( \eta_j \in L^2[0, \infty), \ j = 1, 2. \) Therefore, \( u = iw \) is the ratio of the components of the \( L^2[0, \infty) \) solution, as desired. Since the coefficient functions \( p, v_1 \) and \( v_2 \) are small (in the sense the hypotheses are satisfied), the given system (5.1.1) may be considered a perturbation of (5.3.3). We therefore seek a solution \( u \) of the form (5.2.5), where \( \lim_{x \to \infty} \sum u_n(x, \lambda) = 0. \)

Substitution of (5.2.5) into (5.3.2) gives, omitting function arguments,

\[
\sum_{n=1}^{\infty} u_n' = w^2 v_1 - v_2 - 2iwp - 2ip(\lambda + c + v_1) \sum_{n=1}^{\infty} u_n - (\lambda + c + v_1) \left(u_1^2 + \sum_{n=2}^{\infty} (u_n^2 - 2u_n \sum_{m=1}^{n-1} u_m)\right)
\]

or, upon rearranging

\[
\sum_{n=1}^{\infty} (u_n' + 2i\sqrt{\lambda^2 - c^2} u_n) = w^2 v_1 - v_2 - 2iwp - 2ip(\lambda + c + v_1) \sum_{n=1}^{\infty} u_n - (\lambda^2 + c + v_1) \left(u_1^2 + \sum_{n=1}^{\infty} (u_n + 2u_n \sum_{m=1}^{n-1} u_m)\right).
\]

The sum can be decomposed as

\[
u_1' + 2i\sqrt{\lambda^2 - c^2} u_1 = w^2 v_1 - v_2 - 2iwp \tag{5.3.4}
\]

\[
u_2' + 2i\sqrt{\lambda^2 - c^2} u_2 = -2(p + iv_1)u_1 - (\lambda + c + v_1)u_1^2 \tag{5.3.5}
\]

and for \( n \geq 3, \)

\[
u_n' + 2i\sqrt{\lambda^2 - c^2} u_n = -2(p + iv_1)u_{n-1} - (\lambda + c + v_1) \left(u_{n-1}^2 + 2u_{n-1} \sum_{m=1}^{n-2} u_m\right) \tag{5.3.6}
\]

The solutions \( \{u_n\} \) of (5.3.4)–(5.3.6) are as defined in (5.2.2)–(5.2.4). Lemma 5.3 will show that \( u(x, \lambda) \) as defined in (5.2.5) is both continuous and differentiable.
Lemma 5.2 If $\Im\{\lambda\} \geq 0$ and $|\lambda| > c$, then $\Re\{i\sqrt{\lambda^2 - c^2}\} \leq 0$.

**Proof.** $\Im\{\lambda\} \geq 0$ implies $0 \leq \Arg\{\lambda\} \leq \pi$. If $0 \leq \Arg\{\lambda\} < \pi$, then $0 \leq \Arg\{\lambda^2 - c^2\} < 2\pi$ so $\pi/2 \leq \Arg\{i\sqrt{\lambda^2 - c^2}\} < 3\pi/2$. If $\Arg\{\lambda\} = \pi$, then $\lambda^2 - c^2 = re^{2\pi i}$ for some $r > 0$ and so $\Arg\{i\sqrt{\lambda^2 - c^2}\} = 3\pi/2$. Hence $\pi/2 \leq \Arg\{i\sqrt{\lambda^2 - c^2}\} \geq 3\pi/2$, which is equivalent to $\Re\{i\sqrt{\lambda^2 - c^2}\} \leq 0$. Q.E.D.

Lemma 5.3 If the hypothesis 1–3 are satisfied, then for all $\lambda = \mu + i\epsilon$ with $\epsilon \geq 0$, $|\mu| \geq \Lambda_0$ and for all $x \geq 0$

$$|u_n(x, \lambda)| \leq \frac{a(x)}{2^{n-1}|\sqrt{\lambda^2 - c^2}|}, \quad n = 1, 2, 3, \ldots \tag{5.3.7}$$

and

$$|u'_n(x, \lambda)| \leq \frac{a(x)}{2^{n-3}}, \quad n = 2, 3, 4, \ldots \tag{5.3.8}$$

**Proof.** We use induction on $n$. The case $n = 1$ is true by (5.2.6). For $n = 2$, using the alternate representation for $u_2(x, \lambda)$ in (5.2.9) we have, omitting function arguments,

$$|u_2| \leq \left|\frac{u^2}{2w}\right| + \int_x^\infty |u_1| \left(|wv_1| + \frac{|v_2|}{w} + (|v_1| + 2|\lambda + c|)|u_1|\right) dt$$

$$\leq \frac{a(x)}{2|\sqrt{\lambda^2 - c^2}|} \left[\frac{a(x)}{|\lambda - c|} + 2\int_0^\infty \left(|wv_1| + \frac{|v_2|}{w}\right) dt\right]$$

$$+ 2\int_0^\infty \frac{a(t)}{|w|} dt + \frac{2a(x)}{|\lambda - c|} \int_0^\infty |wv_1| dt$$

$$\leq \frac{a(x)}{2|\sqrt{\lambda^2 - c^2}|} \left[\frac{2}{k_0} + \frac{2}{k_1} + \frac{1}{k_2}\right], \quad |\mu| \geq \Lambda_0$$
using Hypotheses 2 and 3i. Then (5.3.7) follows for $n = 2$. Also, by (5.3.5),

$$|u'_2| \leq |u_1| (2|p| + 2|wv_1| + (|\lambda + c| + |v_1|)|u_1|) + 2\sqrt{\lambda^2 - c^2} |u_2|$$

$$\leq \frac{a(x)}{\sqrt{\lambda^2 - c^2}} \left[ 2|p| + 2|wv_1| + \frac{a(x)}{|w|} + \frac{a(x)|v_1|}{|\sqrt{\lambda^2 - c^2}|} \right] + a(x)$$

and (5.3.8) follows for $n = 2$, $|\mu| \geq \Lambda_0$, using Hypothesis 3ii.

Now assume (5.3.7), (5.3.8) hold for $n = 1, 2, \ldots J$. Then from (5.2.10)

$$|u_{J+1}| \leq \frac{|u_J| |u_J + 2 \sum_{m=1}^{J} u_m|}{2} + \int_x^\infty \left\{ \frac{|u'_J|}{|w|} \sum_{m=1}^{J-1} |u_m| + |u_J| \sum_{m=2}^{J} \frac{|u'_m|}{w} \right\} dt$$

$$+ \frac{a(x)}{2^J \sqrt{\lambda^2 - c^2}} \left\{ 4a(x) \left\{ \frac{4a(x)}{|\lambda - c|} + \int_x^\infty \left( \frac{8a(t)}{|w|} \left( 1 - \frac{1}{2^{J-1}} \right) + \frac{8a(t)}{|w|} \left( 1 - \frac{1}{2^J} \right) \right) 
+ 2 \left( |wv_1| + \frac{|v_2|}{w} + \frac{2a(t)}{|w|} + \frac{4a(t)|v_1|}{\sqrt{\lambda^2 - c^2}} \right) \right\} dt \right\}$$

$$\leq \frac{a(x)}{2^J \sqrt{\lambda^2 - c^2}} \left\{ 20 \int_0^\infty \frac{a(t)}{|w|} dt + 2 \int_0^\infty \left( |wv_1| + \frac{|v_2|}{w} \right) dt 
+ \frac{4a(x)}{|\lambda - c|} \left( 1 + 2 \int_0^\infty |wv_1| dt \right) \right\}$$

$$\leq \frac{a(x)}{2^J \sqrt{\lambda^2 - c^2}} \left\{ \frac{20}{k_0} + \frac{2}{k_1} + \frac{1}{k_2} \right\}. \quad |\mu| \geq \Lambda_0$$

using Hypotheses 2 and 3i. Result (5.3.7) follows. Finally, from (5.3.6),

$$|u'_{J+1}| \leq \frac{2a(x)}{2^{J-1} \sqrt{\lambda^2 - c^2}} \left( |p| + |wv_1| + \frac{2a(x)}{\sqrt{\lambda^2 - c^2}} (|\lambda + c| + |v_1|) \right) + \frac{a(x)}{2^{J-1}}$$
\[
\leq \frac{a(x)}{2J-1} \left( 1 + \frac{2}{|\sqrt{\lambda^2 - c^2}|} \left( |p| + |wv_1| + \frac{2a(x)}{|\sqrt{\lambda^2 - c^2}|}(|\lambda + c| + |v_1|) \right) \right)
\]
\[
\leq \frac{a(x)}{2J-2}, \quad |\mu| \geq \Lambda_0
\]

using Hypothesis 3ii, thus completing the proof. Q.E.D.

Lemma 5.3 shows that \( \sum_{n=1}^{\infty} u_n \) and \( \sum_{n=1}^{\infty} u'_n \) are uniformly absolutely convergent, hence \( u(x, \lambda) \) is indeed a solution to (5.3.2) for \( 0 \leq x < \infty \) and all \( \lambda \) such that \( |\mu| \geq \Lambda_0, \epsilon \geq 0 \).

5.4 Proofs

**Proof of Theorem 5.1.** We show first that \( u \) may be identified with \( \psi_2/\psi_1 \), the ratio of components of an \( L^2[0, \infty) \) solution.

Since \( u(x, \lambda) \) is a solution of the Riccati equation (5.3.2) for \( x \geq 0, |\mu| \geq \Lambda_0, \epsilon \geq 0 \), there exists a nontrivial solution \( y(x, \lambda) = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \) of (5.1.1) such that \( u = y_2/y_1 \), for \( x \geq 0, |\mu| \geq \Lambda_0, \epsilon \geq 0 \). Then using (5.1.1) gives

\[
y'_1 = py_1 + (\lambda + c + v_1)y_2 = (p + (\lambda + c + v_1)u)y_1.
\]

Examining the right side of this equation, we see by Lemma (5.3) and the hypotheses that \( (p + (\lambda + c + v_1)u) \) is finite for all \( x \geq 0 \) and all \( \lambda = \mu + i\epsilon \) with \( |\mu| \geq \Lambda_0, \epsilon \geq 0 \). Also, for given \( \lambda \), \( y \) and \( y' \) cannot simultaneously vanish so we may set

\[
h(x, \lambda) := \frac{y'_1(x, \lambda)}{y_1(x, \lambda)} = p(x) + (\lambda + c + v_1(x))u(x, \lambda).
\]

Then

\[
y_1(x, \lambda) = (\text{constant})e^{\int_0^x h(t, \lambda) \, dt} \tag{5.4.1}
\]
To prove $y_1 \in L^2[0, \infty)$, it suffices to show an $X \in \mathbb{R}$ exists such that for all $x \geq X$, $\text{Re} \left\{ \int_0^x h(t, \lambda) \, dt \right\} < -\delta x$, where $\delta > 0$. We have

$$h(x, \lambda) = p(x) + i\omega v_1(x) + (\lambda + c + v_1(x)) \sum u_n(x, \lambda) + i\sqrt{\lambda^2 - c^2}$$

so

$$\text{Re} \left\{ \int_0^x h(t, \lambda) \, dt \right\} \leq \left| \int_0^x (p(t) + i\omega v_1(t)) \, dt \right| + \left| \int_0^x (\lambda + c + v_1(t)) \sum u_n(t) \, dt \right|

+ \text{Re} \left\{ i\sqrt{\lambda^2 - c^2} \right\} x

\leq \int_0^\infty (|p(t)| + |\omega v_1(t)|) \, dt + \int_0^\infty \frac{|\lambda + c + v_1(t)| |2a(t)|}{|\sqrt{\lambda^2 - c^2}|} \, dt

+ \text{Re} \left\{ i\sqrt{\lambda^2 - c^2} \right\} x$$

where Lemma 5.3 has been used. So

$$\text{Re} \left\{ \int_0^x h(t, \lambda) \, dt \right\} \leq \text{Re} \left\{ i\sqrt{\lambda^2 - c^2} \right\} x + C$$

where $C$ is a constant. Lemma 5.2 shows $\text{Re} \left\{ i\sqrt{\lambda^2 - c^2} \right\} < 0$ for $\text{Im} \{\lambda\} > 0$, so the required $X$ must exist and $y_1 \in L^2[0, \infty)$ for $|\mu| \geq \Lambda_0$, $\epsilon > 0$.

Turning now to $y_2$, since $y_2(x, \lambda) = -\left(\lambda - c + v_2(x)\right)y_1(x, \lambda) - p(x)y_2(x, \lambda)$,

$$y_2(x, \lambda) = \int_0^\infty \exp \left( \int_0^t p(s) \, ds \right) (\lambda - c + v_2(t)) \, y_1(t, \lambda) \, dt$$

which implies, using (5.4.1),

$$|y_2(x, \lambda)| \leq C \int_0^\infty \exp \left( \int_0^\infty |p(s)| \, ds \right) |\lambda - c + v_2(t)| \exp \left( \int_0^t h(s, \lambda) \, ds \right) \, dt

\leq \hat{C} \int_0^\infty \exp \left( \int_0^t h(s, \lambda) \, ds \right) \, dt$$
where

\[ \hat{C} = C \left( \exp \int_0^\infty |p| \right) \sup_{x \leq t \leq \infty} (|\lambda - c + v_2(t)|) < \infty \]

As shown above, the real part of \( \int_0^t h(s, \lambda) \, ds < -\delta t \) for all \( t > X \), so \( y_2(x, \lambda) \in L^2[0, \infty) \), for \( |\mu| \geq \Lambda_0, \epsilon > 0 \).

Thus \( u \) coincides with \( \psi_2/\psi_1 \) for \( \epsilon > 0, |\mu| \geq \Lambda_0 \) and we may use the initial conditions (5.3.1) to write

\[ u(0, \lambda) = \frac{\psi_2(0, \lambda)}{\psi_1(0, \lambda)} = \frac{\sin \alpha + m_\alpha(\lambda) \cos \alpha}{\cos \alpha - m_\alpha(\lambda) \sin \alpha}. \]

Solving for \( m_\alpha(\lambda) \) yields

\[ m_\alpha(\lambda) = \frac{u(0, \lambda) \cos \alpha - \sin \alpha}{u(0, \lambda) \sin \alpha + \cos \alpha}. \]

Since the solution \( u(x, \lambda) \) is also defined for \( \epsilon = 0, |\mu| \geq \Lambda_0 \), the \( \lim_{\epsilon \to 0^+} u(0, \mu + i\epsilon) \) exists and the Titchmarsh-Kodaira formula (1.2.7) may be used. Hence

\[ \rho'_\alpha(\mu) = \frac{1}{\pi} \text{Im} \left\{ \frac{u(0, \mu) \cos \alpha - \sin \alpha}{u(0, \mu) \sin \alpha + \cos \alpha} \right\} \]

for \( |\mu| \geq \Lambda_0 \). We substitute \( u(0, \mu) = S(0, \mu) + iT(0, \mu) \) in the above formula and the result follows.

Q.E.D.

**Proof of the alternate representation** of \( u_n(x, \lambda), n \geq 2 \). From (5.2.3), integration by parts and (5.3.4),

\[
\begin{align*}
    u_2 &= \int_x^\infty e^{2i\sqrt{\lambda^2 - c^2}(t-x)} u_1(2(p + iwv_1) - v_1u_1) \, dt \\
     &\quad - (\lambda + c) \int_x^\infty e^{2i\sqrt{\lambda^2 - c^2}(t-x)} u_1^2 \, dt
\end{align*}
\]
\[
\int_x^\infty e^{2i\sqrt{\lambda^2-c^2}(t-x)} u_1(2(p + iwv_1) - v_1 u_1) \, dt + \frac{u_1^2}{2iw} \\
+ \frac{1}{iw} \int_x^\infty e^{2i\sqrt{\lambda^2-c^2}(t-x)} u_1(w^2 v_1 - v_2 - 2iw p - 2i\sqrt{\lambda^2-c^2} u_1) \, dt \\
= \frac{u_1^2}{2iw} + \int_x^\infty e^{2i\sqrt{\lambda^2-c^2}(t-x)} (iwv_1 u_1 - \frac{v_2 u_1}{iw} - (v_1 + 2(\lambda + c)) u_1^2) \, dt
\]
and \((5.2.9)\) follows. Similarly, from \((5.2.4)\) and integration by parts.

Here, only \(u_1\) is replaced by \((5.3.4)\) and the simplified expression yields \((5.2.10)\).

### 5.5 Examination of Conditions for which Theorem 5.1 Holds

The two lemmas 5.4 and 5.5 show that \(a(x)\) in Hypothesis 1 can be computed when certain restrictions are placed on the coefficient functions \(p, v_1\) and \(v_2\).

**Lemma 5.4** If \(v_1, v_2, p\) and their derivatives are each monotonic \(L^1[0, \infty)\) functions, then there exists a decreasing \(L^1[0, \infty)\) function \(a(x)\) such that \(|u_1(x, \lambda)| \leq \frac{a(x)}{\sqrt{\lambda^2 - c^2}}\).

**Proof.** Write \(f(t) = w^2 v_1(t) - v_2(t) - 2iw p(t)\). Integrating \((5.2.2)\) by parts yields

\[
u_1(x, \lambda) = \frac{-f(x)}{2i\sqrt{\lambda^2 - c^2}} - \frac{1}{2i\sqrt{\lambda^2 - c^2}} \int_x^\infty f'(t) e^{2i\sqrt{\lambda^2-c^2}(t-x)} \, dt
\]
\[
|u_1(x, \lambda)| \leq \frac{1}{2|\sqrt{\lambda^2 - c^2}|} \left[ |f(x)| + |w^2v_1(x)| + |v_2(x)| + 2|wp(x)| \right]
\]

where the monotonicity of the \(L^1\) functions has been used. So

\[
|u_1(x, \lambda)| \leq \frac{1}{|\sqrt{\lambda^2 - c^2}|} \left( |w^2v_1(x)| + |v_2(x)| + 2|wp(x)| \right). 
\]

Since \(|w| \to 1\) as \(|\lambda| \to \infty\), we may define

\[
a(x) := \sup_{|\lambda| \geq e} \{ |w^2v_1(x)| + |v_2(x)| + 2|wp(x)| \}.
\]

Note: A smaller \(a(x)\) may be obtained in an iterative process by taking the supremum over the set \(\{\lambda = \mu + i\epsilon : |\mu| \geq \Lambda_0, \Lambda_0\}\) as in Hypothesis 3. Q.E.D.

**Lemma 5.5** If \(v_1, v_2\) and \(p\) are each decreasing \(L^1[0, \infty)\) functions and \(\lambda = \mu + i\epsilon\) satisfies \(|\mu| \geq c, 0 \leq \epsilon \leq \sqrt{\mu^2 - c^2}\), then there exists a decreasing \(L^1[0, \infty)\) function \(a(x)\) such that \(|u_1(x, \lambda)| \leq \frac{a(x)}{|\sqrt{\lambda^2 - c^2}|} \).

The proof of Lemma 5.5 follows Lemmas 5.6 and 5.7.

**Lemma 5.6** The conditions \(|\mu| \geq c\) and \(0 \leq \epsilon \leq \sqrt{\mu^2 - c^2}\) imply \(|Re\{\sqrt{\lambda^2 - c^2}\}| \geq Im\{\sqrt{\lambda^2 - c^2}\} \geq 0\) where \(\lambda = \mu + i\epsilon\).

**Proof.** The proof of Lemma 5.2 shows \(Im\{\sqrt{\lambda^2 - c^2}\} \geq 0\) for \(\epsilon \geq 0\). Also \(0 \leq \epsilon \leq \sqrt{\mu^2 - c^2}\) implies \(0 \leq \mu^2 - c^2 - \epsilon^2 = Re\{\lambda^2 - c^2\}\). Geometrically, this means \(\lambda^2 - c^2\) is in the first or fourth quadrant of the complex plane. Hence

\[
Arg\{\lambda^2 - c^2\} \in [0, \pi/2] \cup [3\pi/2, 2\pi)
\]

\[
Arg\{\sqrt{\lambda^2 - c^2}\} \in [0, \pi/4] \cup [3\pi/4, \pi).
\]
In the case \( \lambda = \mu < -c \), then \( \lambda^2 - c^2 = re^{2\pi i} \), some \( r > 0 \), and so \( \text{Arg}\{\sqrt{\lambda^2 - c^2}\} = \pi \).

Arguments in these regions of the complex plane are equivalent to \( |\text{Re}\{\sqrt{\lambda^2 - c^2}\}| \geq \text{Im}\{\sqrt{\lambda^2 - c^2}\} \).

Q.E.D.

Note: The above proof is reversible, so the conditions in the lemma are equivalent.

**Lemma 5.7** If \( \text{Im}\{\lambda\} \geq 0 \), then \( 0 \leq \text{Arg}\{w\} \leq \pi/2 \). That is, \( w = \sqrt{\frac{\lambda - c}{\lambda + c}} \) falls in the first quadrant of the complex plane.

**Proof.** The Möbius transformation \( \frac{z - c}{z + c}, c \) real, maps the upper half plane onto itself. Given \( \text{Im}\{\lambda\} \geq 0 \), then \( 0 \leq \text{Arg}\{\lambda\} \leq \pi \) which implies

\[
0 \leq \text{Arg}\left\{\frac{\lambda - c}{\lambda + c}\right\} \leq \pi
\]

\[
0 \leq \text{Arg}\left\{\sqrt{\frac{\lambda - c}{\lambda + c}}\right\} \leq \pi/2
\]

using the square root as defined in Chapter 1.

Q.E.D.

**Proof of Lemma 5.5.** We write \( \sqrt{\lambda^2 - c^2} = d + bi \) and \( w = |w|e^{i\tau} \). For \( \text{Im}\{\lambda\} \geq 0 \), we have \( b \geq 0 \) (by proof of Lemma 5.2) and \( 0 \leq \tau \leq \pi/2 \) (by Lemma 5.7). Using this notation, (5.2.2) can be written

\[
u_1(x, \lambda) = -\int_x^\infty e^{-2b(t-x)} \left( |w|^2 v_1 e^{i(2d(t-x) + 2\tau)} - v_2 e^{i(2d(t-x))} \right. \\
- \left. 2 |w| p e^{i(2d(t-x) + \tau + \pi/2)} \right) dt.
\]
So

\[
Re\{u_1\} = - \int_\infty^\xi e^{-2b(t-x)} \left( |w|^2 v_1 \cos(2d(t-x) + 2\tau) - v_2 \cos(2d(t-x)) \right) \\
- 2|w|p \cos(2d(t-x) + \tau + \pi/2) \right) \, dt
\]

and \(Im\{u_1\}\) is similar. Using the Second Mean Value Theorem [22]

\[
Re\{u_1\} = -v_1(x) \int_x^{\xi_1} |w|^2 \cos(2d(t-x) + 2\tau) \, dt + v_2(x) \int_x^{\xi_2} \cos(2d(t-x)) \, dt \\
+ 2p(x) \int_x^{\xi_3} |w| \cos(2d(t-x) + \tau + \pi/2) \, dt
\]

for \(\xi_i \in [x, \infty), i = 1, 2, 3\). Therefore

\[
Re\{u_1\} = -\frac{|w|^2 v_1}{2d} (\sin(2d(\xi_1 - x) + 2\tau) - \sin 2\tau) \\
+ \frac{v_2}{2d} \sin 2d(\xi_2 - x) + \frac{|w|p}{d} (\sin(2d(\xi_3 - x) + \tau + \pi/2) - \sin(\tau + \pi/2)) .
\]

The term \(\sin(\tau + \pi/2)\) is nonnegative so

\[
|Re\{u_1\}| \leq \frac{|w^2 v_1| + |v_2| + |wp|}{|d|} = \frac{|w^2 v_1| + |v_2| + |wp|}{|Re\{\sqrt{\lambda^2 - c^2}\}|} .
\]

Similarly

\[
|Im\{u_1\}| \leq \frac{|w^2 v_1| + |v_2| + |wp|}{|Re\{\sqrt{\lambda^2 - c^2}\}|}
\]

so

\[
|u_1| \leq 2\frac{(|w^2 v_1| + |v_2| + |wp|)}{|Re\{\sqrt{\lambda^2 - c^2}\}} .
\]

By Lemma 5.6,

\[
|Re\{\sqrt{\lambda^2 - c^2}\}| \geq Im\{\sqrt{\lambda^2 - c^2}\} .
\]
hence

\[ |\sqrt{\lambda^2 - c^2}| \leq \sqrt{2} |\text{Re}\{\sqrt{\lambda^2 - c^2}\}| \]
\[ |u_1| \leq \frac{2\sqrt{2}(|w^2v_1| + |v_2| + |wp|)}{|\sqrt{\lambda^2 - c^2}|}. \]

We therefore define

\[ a(x) := \sup_{|\lambda| \geq c} \{2\sqrt{2}(|w^2v_1(x)| + |v_2(x)| + |wp(x)|)\}. \]

Note: If \( \epsilon \) is forced to be smaller, say by requiring \( |\text{Re}\{\sqrt{\lambda^2 - c^2}\}| \geq k\text{Im}\{\sqrt{\lambda^2 - c^2}\}, \)

\( k > 1, \) then \( a(x) \) may be improved by replacing the \( \sqrt{2} \) by \( (1 + \delta) \) for \( \delta > 0 \) as small as desired. Additionally, a smaller \( a(x) \) is achieved by taking the supremum over the set

\[ \{\lambda = \mu + i\epsilon : |\mu| \geq \Lambda_0, \Lambda_0 \text{ as in Hypothesis 3}\}. \]

Q.E.D.

5.6 Examples

1. We begin by considering the case \( p \equiv v_1 \equiv v_2 \equiv 0. \) It is easy to verify that \( u(x, \lambda) = iw \) is the solution to the Riccati equation (5.3.2) of the form (5.2.5). For \( \lambda = \mu + i\epsilon \) with \( |\mu| > c, \)

\[ S(x, \mu + i\epsilon) \equiv 0 \]
\[ T(x, \mu + i\epsilon) \rightarrow \sqrt{\frac{\mu - c}{\mu + c}} \text{ as } \epsilon \rightarrow 0^+ \]
Therefore

\[ \rho'_\alpha(\mu) = \frac{1}{\pi} \frac{\sqrt{\mu - c}}{(\mu + c) \sin^2 \alpha + \cos^2 \alpha} \]

\[ = \begin{cases} 
\frac{\sqrt{\mu^2 - c^2}}{\pi (\mu + c \cos 2\alpha)} & \mu > c \\
\frac{-\sqrt{\mu^2 - c^2}}{\pi (\mu + c \cos 2\alpha)} & \mu < -c
\end{cases} \]

which agrees with Theorem 5.1, where the hypotheses are satisfied by \( a(x) = 0 \) and any \( \Lambda_0 > c \).

2. We now consider the case \( p \equiv 0, v_1(x) = v_2(x) = \frac{1}{k} e^{-x} \). Using (5.2.2),

\[ u_1(x, \lambda) = \frac{(w^2 - 1)e^{-x}}{k(2i\sqrt{\lambda^2 - c^2} - 1)} \]

whence

\[ |u_1(x, \lambda)| = \frac{2c|w|e^{-x}}{k|2i\sqrt{\lambda^2 - c^2} - 1|} \cdot \frac{1}{\sqrt{\lambda^2 - c^2}} \leq \frac{2c|w|e^{-x}}{k(2|\sqrt{\lambda^2 - c^2}| - 1)} \cdot \frac{1}{\sqrt{\lambda^2 - c^2}} \]

Note that if \( c = 0 \), then \( w \equiv 1 \) for \( \lambda \neq 0 \) and \( u_1(x, \lambda) = 0 \). Hence, \( u(x, \lambda) = i \), and so for all \( \mu \neq 0 \), \( \rho'_\alpha(\mu) = 1/\pi \) for all \( \alpha \in [0, \pi) \) and any \( k > 0 \). For \( c \neq 0 \), we claim that for \( k > 4 \), the hypotheses are satisfied. It is convenient to set

\[ a(x, \lambda) = \frac{2c|w|e^{-x}}{k(2|\sqrt{\lambda^2 - c^2}| - 1)} \]

rather than take \( a \) as a function of \( x \) only. Then the integral in Hypothesis 2ii is \( \frac{2c}{k(2|\sqrt{\lambda^2 - c^2}| - 1)} \) and for any \( k_0 \), a \( \Lambda_0 \) can be chosen so that the inequality is
satisfied for all $|\mu| \geq \Lambda_0$. From Hypothesis (2iii), it is necessary that

$$\frac{1}{k}(|w| + |w|^{-1}) < \frac{1}{k_1}.$$ 

Since $|w| \to 1$ as $|\lambda| \to \infty$ and since Hypothesis (2i) requires $k_1 > 2$, this condition is satisfied for $|\lambda|$ large enough as long as $k > 4$. Given such a $k$, we choose $k_1$ satisfying $2 < k_1 < k/2$ and $k_0, k_2 > 0$ so that

$$\frac{20}{k_0} + \frac{1}{k_2} \leq 1 - \frac{2}{k_1}.$$ 

We set $\Lambda_0 > c$ so large that each of the following are satisfied for all $|\mu| \geq \Lambda_0$:

$$|\sqrt{\lambda^2 - c^2}| > \frac{ck_0}{k} + \frac{1}{2} \quad \text{... to satisfy (2ii)}$$

$$\frac{2k_1}{k} \leq |w| \leq \frac{k}{2k_1} \quad \text{... to satisfy (2iii)}$$

$$|\lambda - c| / a(0, \lambda) \geq 4k_2 \quad \text{... to satisfy (3i)}$$

$$|\sqrt{\lambda^2 - c^2}| > \frac{2k_2 + 1}{2k_1 k_2} \quad \text{... to satisfy (3ii)}$$

As hypotheses 1–3 are satisfied, Theorem 5.1 applies for this Dirac equation for all $|\mu| \geq \Lambda_0$. For example, if $k = 8$ and $c = 1$, valid choices for $k_i, i = 0, 1, 2$ are $k_0 = 240$, $k_1 = 3$, and $k_2 = 4$. In this case, $\Lambda_0 = 31$.

3. As a final example, we consider the case $v_1 \equiv v_2 \equiv 0$ and $p(x) = \frac{1}{k(x + 1)^2}$ where $k > 20$. Using (5.2.2),

$$u_1(x, \lambda) = \frac{2iw}{k} \int_x^\infty \frac{e^{2i\sqrt{\lambda^2 - c^2(t-x)}(t+1)^{-2}}}{t} \, dt$$
and integration by parts shows

$$|u_1(x, \lambda)| \leq \frac{|w|}{k} \frac{(x + 1)^{-2}}{\sqrt{\lambda^2 - c^2}}.$$

We therefore take

$$a(x, \lambda) := \frac{|w|(x + 1)^{-2}}{k}.$$

Here, allowing $a$ to be a function of $x$ and $\lambda$ simplifies calculations; however, as $|w|$ is bounded, it could be replaced by a constant. Then, from Hypothesis 2ii,

$$\int_0^\infty \frac{a(x, \lambda)}{|w|} = \frac{1}{k} < \frac{1}{k_0}.$$

From Hypothesis 2i, $\frac{20}{k_0} < 1$, so this establishes the requirement $k > 20$. For any $k > 20$, choose $k_0$ so that $20 < k_0 < k$ and choose $k_2$ so that $\frac{20}{k_0} + \frac{1}{k_2} \leq 1$. The third hypothesis is satisfied by taking $\Lambda_0$ so large that for all $|\mu| \geq \Lambda_0$,

$$|\sqrt{\lambda^2 - c^2}| > \frac{4k_2}{k}.$$

If $k = 22$, a valid choice for both $k_0$ and $k_2$ is 21. Then all hypotheses are satisfied for any $\Lambda_0 > 84/22$ and the $\rho'_\alpha(\mu)$ for this Dirac equation are given by Theorem 5.1 for $|\mu| \geq \Lambda_0$. 
CHAPTER 6

A REFINEMENT OF THE DIRAC EQUATION

CONNECTION FORMULA

6.1 Introduction

This chapter refines a result of Chapter 4 concerning the connections among the spectral derivatives \( \rho'_\alpha(\mu), \mu \in \mathbb{R} \), associated with the limit point Dirac equation

\[
y' = \begin{pmatrix} p & \lambda + c + v_1 \\ - (\lambda - c + v_2) & -p \end{pmatrix} y, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}
\]

(6.1.1)

on \([0, \infty)\) together with the initial condition

\[
y_1(0) \cos \alpha + y_2(0) \sin \alpha = 0
\]

(6.1.2)

where \( \alpha \in [0, \pi) \). Also, \( c \geq 0 \) is a constant, \( \lambda = \mu + i \epsilon \) is the complex spectral parameter and \( p, v_1, v_2 \) are bounded, real valued functions satisfying the additional hypotheses listed below. To each parameter \( \alpha \), there is associated the unique Titchmarsh-Weyl \( m_\alpha(\lambda) \) function which is defined by

\[
\theta_\alpha(x, \lambda) + m_\alpha(\lambda) \varphi_\alpha(x, \lambda) \in L^2[0, \infty)
\]

(6.1.3)

where \( \theta_\alpha, \varphi_\alpha \) are solutions of (6.1.1) which satisfy for all \( \lambda \)

\[
\theta_\alpha(0, \lambda) = \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}, \quad \varphi_\alpha(0, \lambda) = \begin{pmatrix} - \sin \alpha \\ \cos \alpha \end{pmatrix}.
\]

(6.1.4)
We take the coefficient functions $p, v_1$, and $v_2$ such that the spectral functions $\rho'_\alpha(\mu)$ associated with (6.1.1) satisfy these conditions referred to as (A):

(i) There is a $\Lambda_0 \in \mathbb{R}$ such that for all $|\mu| \geq \Lambda_0$, $\rho'_\alpha(\mu)$ is continuous and $0 < \rho'_\alpha(\mu) < \infty$ for all $\alpha \in [0, \pi)$.

(ii) There exist real valued functions $S(\mu), T(\mu)$ such that for $\mu \geq \Lambda_0$,

$$
\rho'_\alpha(\mu) = \frac{1}{\pi} \frac{T(\mu)}{(S(\mu)^2 + T(\mu)^2) \sin^2 \alpha + S(\mu) \sin 2\alpha + \cos^2 \alpha}
$$

(6.1.5)

with $S(\mu) \to 0$ and $T(\mu) \to \sqrt{\frac{\mu - c}{\mu + c}}$ as $|\mu| \to \infty$.

These conditions are met, for example, by Dirac equations which satisfy hypotheses (1)–(3) of Chapter 5. From (6.1.5), we see $\rho'_\alpha(\mu) \to 1/\pi$ for all $\alpha \in [0, \pi)$.

Theorem 4.2 established in Chapter 4 for a more general Dirac equation states that given $\rho'_\alpha(\mu)$ for fixed $\mu$ and two distinct values of $\alpha$, a third derivative must be one of two choices. In the restricted class of equations considered here, this result is improved. Using the additional information on the asymptotic behavior of the spectral derivatives, it is often possible to distinguish the correct value of $\rho'_\alpha(\mu)$ from the two possibilities. The result is as follows.

**Theorem 6.1** For the Dirac equation (6.1.1) satisfying the conditions (A), distinct $\alpha, \beta, \gamma \in [0, \pi)$, and $|\mu| \geq \Lambda_0 > c$,

(i) if $\cos(\beta - \alpha) > 0$,

$$
\frac{\sin^2(\beta - \alpha)}{\rho'_\gamma(\mu)} = \frac{\sin^2(\beta - \gamma)}{\rho'_\alpha(\mu)} + \frac{\sin^2(\gamma - \alpha)}{\rho'_\beta(\mu)}
$$

$$
+ 2 \sin(\beta - \gamma) \sin(\gamma - \alpha) \sqrt{\frac{1}{\rho'_\alpha(\mu) \rho'_\beta(\mu)}} - \pi^2 \sin^2(\beta - \alpha).
$$

(6.1.6)
(ii) if $\cos(\beta - \alpha) < 0$,

$$\frac{\sin^2(\beta - \alpha)}{\rho'_\gamma(\mu)} = \frac{\sin^2(\beta - \gamma)}{\rho'_\alpha(\mu)} + \frac{\sin^2(\gamma - \alpha)}{\rho'_\beta(\mu)}$$

$$- 2 \sin (\beta - \gamma) \sin (\gamma - \alpha) \left[ \frac{1}{\rho'_\alpha(\mu)\rho'_\beta(\mu)} - \pi^2 \sin^2 (\beta - \alpha) \right]. \quad (6.1.7)$$

The theorem omits the case where $\alpha$ and $\beta$ differ by $\pi/2$, for then the two choices are asymptotically indistinguishable. However, if additional information is available on $S(\mu)$, the theorem can be extended to this case. The theorem is proved in §6.2 and examples follow in §6.3.

6.2 Proof of Theorem 6.1

The two possible values for $\rho'_\alpha(\mu)$ are given in Theorem 4.2. These expressions may be written in terms of $S(\mu), T(\mu)$ using (6.1.5). That is, the two choices for

$$\frac{\sin^2(\beta - \alpha)}{\rho'_\gamma(\mu)}$$

are

$$\frac{\pi}{T} \left\{ \left( (S^2 + T^2) \sin^2 \alpha + S \sin 2\alpha + \cos^2 \alpha \right) \sin^2 (\beta - \gamma) \right.$$

$$\left. + \left( (S^2 + T^2) \sin^2 \beta + S \sin 2\beta + \cos^2 \beta \right) \sin^2 (\gamma - \alpha) \right. \right.$$  

$$\left. \pm 2 \sin(\beta - \gamma) \sin(\gamma - \alpha) \left| (S^2 + T^2) \sin \alpha \sin \beta + S \sin(\alpha + \beta) + \cos \alpha \cos \beta \right| \right\}. \quad (6.2.1)$$

We assume for the present that the expression within absolute value is nonnegative and consider the choice using the positive sign. Then we have

$$\frac{\pi}{T} \left\{ (S^2 + T^2) \left( \sin \alpha \sin(\beta - \gamma) + \sin \beta \sin(\gamma - \alpha) \right)^2 + \right.$$

$$S \left( \sin 2\alpha \sin^2(\beta - \gamma) + \sin 2\beta \sin^2(\gamma - \alpha) + 2 \sin(\beta - \gamma) \sin(\gamma - \alpha) \sin(\alpha + \beta) \right) \right.$$
\[
\left\{ (S^2 + T^2) \sin^2 \gamma + S \sin 2\gamma + \cos^2 \gamma \right\}
\]

which upon applying trigonometric identities as in §3.2 simplifies to

\[
\frac{\pi \sin^2(\beta - \alpha)}{T} \left\{ (S^2 + T^2) \sin^2 \gamma + S \sin 2\gamma + \cos^2 \gamma \right\}.
\]

This is identically \( \frac{\sin^2(\beta - \alpha)}{\rho'_\gamma(\mu)} \). If the expression within absolute value is negative, then the choice using the negative sign is the one that simplifies as above. Recall that the spectral derivatives \( \rho'_\gamma(\mu) \) for the restricted class of Dirac equations are continuous on \( \mu \geq \Lambda_0 \) and on \( \mu \leq -\Lambda_0 \). Therefore, \( \rho'_\gamma(\mu) \) for two \( \mu \) on the same half line are both given by either the choice using the positive sign or the choice using the negative sign. We can thus use the asymptotic behavior of \( S(\mu) \) and \( T(\mu) \) to choose the correct sign for large \( |\mu| \) and be assured the choice is correct for all \( |\mu| \geq \Lambda_0 \). In fact, since

\[
\left| \left( S(\mu)^2 + T(\mu)^2 \right) \sin \alpha \sin \beta + S(\mu) \sin(\alpha + \beta) + \cos \alpha \cos \beta \right| \to |\cos(\beta - \alpha)|,
\]

formulae (6.1.6) and (6.1.7) are established. But if \( |\beta - \alpha| = \pi/2 \), then the quantity within absolute value tends to 0 and more information on the size and sign of \( S(\mu) \) is required to determine whether the approach is from above or below. In general, we may only conclude

\[
\frac{\sin^2(\beta - \alpha)}{\rho'_\gamma(\mu)} \rightarrow \frac{\sin^2(\alpha + \pi/2 - \gamma)}{\rho'_\alpha(\mu)} + \frac{\sin^2(\gamma - \alpha)}{\rho'_{\alpha+\pi/2}(\mu)} \text{ as } |\mu| \to \infty.
\]

6.3 Examples

Consider the case \( p \equiv v_1 \equiv v_2 \equiv 0 \). Then the conditions (A) are satisfied and Theorem 6.1 applies for \( |\mu| \geq c \). The quantities \( S(\mu), T(\mu) \) and \( \rho'_\alpha(\mu) \) can be
explicitly computed and are given by

\[ S(\mu) \equiv 0, \quad T(\mu) = \frac{\sqrt{\mu - c}}{\mu + c} \]

\[ \rho'_\alpha(\mu) = \begin{cases} \frac{\sqrt{\mu^2 - c^2}}{\pi(\mu + c \cos 2\alpha)} & \mu > c \\ \frac{-\sqrt{\mu^2 - c^2}}{\pi(\mu + c \cos 2\alpha)} & \mu < -c \end{cases} \]

Suppose \( \alpha = \frac{\pi}{4}, \beta = \frac{\pi}{2} \) so that for \( \mu > c \)

\[ \rho'_{\pi/4}(\mu) = \frac{\sqrt{\mu^2 - c^2}}{\pi \mu} \quad \text{and} \quad \rho'_{\pi/2}(\mu) = \frac{\sqrt{\mu^2 - c^2}}{\pi(\mu - c)} \]

Formula (6.1.6) applies and gives for \( \gamma = 0 \)

\[ \frac{1}{\rho_0(\mu)} = \pi \left[ \frac{3\mu/2 - c/2}{\sqrt{\mu^2 - c^2}} + 2(1 - \frac{1}{\sqrt{2}}) \sqrt{\frac{1}{2} \mu^2 - c^2} \right] \]

\[ = \frac{\pi}{2} \left[ \frac{\mu + c}{\sqrt{\mu^2 - c^2}} \right]. \]

So \( \rho'_0(\mu) = \frac{1}{\pi} \frac{\sqrt{\mu^2 - c^2}}{\mu + c} \), for \( \mu > c \), as expected. The rejected value is \( \frac{1}{\pi} \frac{\sqrt{\mu^2 - c^2}}{5\mu - 3c} \).

Note that the chosen value tends to \( \frac{1}{\pi} \) as \( \mu \to \infty \) but that the rejected value does not have the correct asymptotic behavior. The calculations are similar for \( \mu < -c \).

Suppose now \( \alpha, \beta \) differ by \( \pi/2 \). Writing \( \beta = \alpha + \pi/2 \) and neglecting the absolute value, both expressions in (6.2.1) simplify and are \( \frac{\sin^2(\beta - \alpha)}{\rho'_\alpha(\mu)} \) and \( \frac{\sin^2(\beta - \alpha)}{\rho'_{2\alpha-\gamma}(\mu)} \). In other words, both the positive and negative branches of the square root yield valid expressions for spectral derivatives. As a specific example, we set \( p \equiv v_1 \equiv v_2 \equiv 0 \) and \( \alpha = \pi/4, \beta = 3\pi/4 \). Then if \( \gamma = 0 \), the choices for \( \frac{\sin^2(\beta - \alpha)}{\rho'_0(\mu)} \) for \( \mu > c \) are

\[ \pi \left[ \frac{\mu}{\sqrt{\mu^2 - c^2}} \pm 2 \left( \frac{1}{\sqrt{2}} \right) \left( \frac{-1}{\sqrt{2}} \right) \sqrt{\frac{c^2}{\mu^2 - c^2}} \right]. \]
That is, for $\mu > c$, $\rho'_0(\mu)$ is either $\frac{\sqrt{\mu^2 - c^2}}{\pi(\mu - c)}$ or $\frac{\sqrt{\mu^2 - c^2}}{\pi(\mu + c)}$. The expression that is not $\rho'_0(\mu)$ is $\rho'_{2\alpha-\gamma}(\mu) = \rho'_{\pi/2}(\mu)$. Asymptotically, these are indistinguishable which is why the proof of the theorem fails in this situation. However, in this simple case, since $S(\mu)$ and $T(\mu)$ are explicitly known, the expression within absolute value in (6.2.1) can be evaluated. In fact,

\[
\left| (S^2 + T^2) \sin \alpha \sin(\alpha + \pi/2) + S \sin(2\alpha + \pi/2) + \cos \alpha \cos(\alpha + \pi/2) \right|
\]

\[
= \left| \frac{\mu - c}{\mu + c} \sin \alpha \cos \alpha - \sin \alpha \cos \alpha \right|
\]

\[
= \left| \frac{-2c}{\mu + c} \sin \alpha \cos \alpha \right|.
\]

Since $c$ is nonnegative and $\alpha$ is in the first quadrant, this quantity is nonpositive for $\mu > c$ and so (6.1.7) is used and yields $\rho'_0(\mu) = \frac{\sqrt{\mu^2 - c^2}}{\pi(\mu - c)}$. For $\mu < -c$, the quantity is nonnegative and so (6.1.6) must be used and $\rho'_0(\mu) = \frac{\sqrt{\mu^2 - c^2}}{\pi(-\mu - c)} = \frac{-\sqrt{\mu^2 - c^2}}{\pi(\mu + c)}$ is easily calculated.
REFERENCES


