Homework Problems on Topological Spaces

1. Prove DeMorgan’s laws: \((\bigcup A_a)^c = \bigcap A_a^c\) and \((\bigcap A_a)^c = \bigcup A_a^c\).

2. A) Let \(\mathcal{F}_c\) be the collection of all subsets \(U\) of \(X\) such that \(X - U\) is countable or all of \(X\). Show that \(\mathcal{F}_c\) is a topology on \(X\).
   B) Is the collection of all subsets \(U\) of \(X\) such that \(X - U\) is infinite, empty, or all of \(X\) a topology?

3. Suppose \(\{\mathcal{F}_i\}\) is a collection of topologies on \(X\).
   A) Show that the intersection of all the \(\mathcal{F}_i\) is a topology.
   B) Is the union of all the \(\mathcal{F}_i\) a topology?
   C) Show that there is a unique smallest topology containing all the \(\mathcal{F}_i\) and a unique largest topology contained in all of the \(\mathcal{F}_i\).

4. Let \(X = \{a, b, c\}\), \(\tau = \{X, \{\}, \{a\}, \{a, b\}\}\) and \(\tau' = \{X, \{\}, \{a\}, \{b, c\}\}\).
   A) Find the smallest topology containing \(\tau\) and \(\tau'\).
   B) Find the largest topology contained in \(\tau\) and \(\tau'\).

5. Let \(X\) be a topological space. Show that the following conditions hold:
   A) The empty set and \(X\) are closed.
   B) Arbitrary intersections of closed sets are closed.
   C) Finite unions of closed sets are closed.

6. Prove the following:
   A) \(A \cup B = \overline{A} \cup \overline{B}\)
   B) \(\bigcup A_a \supset \bigcup A_a\) (Give an example where equality fails.)

7. Let \(\tau\) and \(\tau'\) be two topologies on a set \(X\) and let \(i: (X, \tau') \to (X, \tau)\) be the identity map.
   A) \(\tau'\) is finer than \(\tau\) \(\iff\) \(i\) is continuous.
   B) \(\tau' = \tau \iff\) \(i\) is a homeomorphism.

8. Let \(\mathcal{F}_n\) be the topology on the real line generated by the usual basis plus \(\{n\}\). Show that \((R, \mathcal{F}_1)\) and \((R, \mathcal{F}_2)\) are homeomorphic, but that \(\mathcal{F}_1\) does not equal \(\mathcal{F}_2\).

9. Find a function from \(R\) to \(R\) that is continuous at precisely one point.

10. Show that if \(X \subset Y \subset Z\) then the subspace topology on \(X\) as a subspace on \(Y\) is the same as the subspace topology on \(X\) as a subspace of \(Z\).

11. Show that the countable collection \(\{(a, b) \times (c, d)\} \mid a < b\) and \(c < d\) and \(a, b, c, d\) are rational\} is a basis for \(R^2\).
12. Determine which of the following equations hold. If not, determine whether any inclusion holds.
   A) \( A \cap B = \overline{A} \cap \overline{B} \)
   B) \( \bigcap A_a = \bigcap \overline{A}_a \)
   C) \( A - B = \overline{A} - \overline{B} \)
   D) \( (A \cup B)' = A' \cup B' \)
   E) \( (A \cap B)' = A' \cap B' \)

13. If \( \mathcal{S}_1 \) is finer than \( \mathcal{S}_2 \), what does the connectedness of \( X \) in one topology imply about the connectedness of \( X \) in the other?

14. Let \( \{A_n\} \) be a sequence of connected sets such that \( A_n \) intersects \( A_{n+1} \) nontrivially for each \( n \). Show that \( \bigcup A_n \) is connected.

15. Show that if \( X \) is an infinite set then it is connected in the finite complement topology.

16. If \( \mathcal{S}_1 \) is finer than \( \mathcal{S}_2 \), what does the compactness of \( X \) in one topology imply about the compactness of \( X \) in the other?

17. Show the following:
   (a) \( \text{Bd} (A) \) is empty iff \( A \) is both open and closed.
   (b) \( A \) is open iff \( \text{Bd} (A) = \overline{A} - A \).

18. For any subset \( A \) of the real line (with the usual topology) there are at most 14 sets (including \( A \)) that can be formed by using complementation and closure. Prove this by completing the following steps:
   (a) Show that if \( A \) is open then \( A = A^{-c-c-c} \).
   (b) Let \( K = \{ A, \overline{A}, A^{-c}, A^{-c-c}, A^{-c-c-c}, A^{-c-c-c-c}, A^{-c-c-c-c-c}, A^{-c-c-c-c-c} \} \).
      Show that \( K \) is closed under complementation and closure.
   (c) Show that there is an \( A \) such that \( K \) has exactly 14 distinct elements.

19. For any subset \( A \) of the real line (with the usual topology) there are at most 7 sets (including \( A \)) that can be formed by using the interior and closure operations. Prove this by completing the following steps:
   (a) Show that \( A^{-o} = A^{o-o-o} \) and \( A^{-o} = A^{-o-o} \).
   (b) Let \( K = \{ A, \overline{A}, A^{-o}, A^{-o-o}, A^{-o-o-o}, A^{-o-o-o-o} \} \).
      Show that \( K \) is closed under the interior and closure operations.
   (c) Show that there is an \( A \) such that \( K \) has exactly 7 distinct elements.

20. Show that the product of two Hausdorff spaces is Hausdorff.
21. Prove the Extreme Value Theorem: Let \( f : X \to Y \) be a continuous map of a compact space \( X \) into an ordered space \( Y \) (with the order topology). Then there are points \( a \) and \( b \) in \( X \) such that \( f(a) \leq f(x) \leq f(b) \) for every \( x \) in \( X \).

22. Prove the following: A topological space \( X \) is compact iff every collection \( \{ C_{a} \} \) of closed subsets of \( X \) with the property that the intersection of any finite subcollection is nonempty also has the property that \( \bigcap C_{a} \) is nonempty.

Homework Problems on Metric Spaces

1. In \( \mathbb{R}^n \), define \( d(x, y) = |x_1 - y_1| + \ldots + |x_n - y_n| \). Show that \( d \) is a metric that induces the usual topology. Sketch the basis elements when \( n = 2 \).

2. In \( \mathbb{R}^n \), for \( p \geq 1 \) define \( d(x, y) = \sum_{i=1}^{n} (|x_i - y_i|^p)^{1/p} \). Assume that \( d \) is a metric. Show that it induces the usual topology.

3. Show that the topology induced by a metric is the coarsest topology relative to which the metric is continuous.

4. Let \( d \) be a metric. Show that \( d'(x, y) = d(x, y)/(1 + d(x, y)) \) is a bounded metric.

5. Let \( d \) be a metric. Show that \( \overline{d}(x, y) = \min\{d(x, y), 1\} \) induces the same topology as \( d \).

6. For \( x \) and \( y \) in \( \mathbb{R}^n \), let \( x \cdot y = \sum x_i y_i \) and \( \|x\| = \sqrt{x \cdot x} \). Show that the Euclidean metric \( d \) on \( \mathbb{R}^n \) is a metric by completing the following:
   (a) Show that \( x \cdot (y + z) = (x \cdot y) + (x \cdot z) \).
   (b) Show that \( |x \cdot y| \leq \|x\| \|y\| \).
   (c) Show that \( \|x + y\| \leq \|x\| + \|y\| \).
   (d) Verify that \( d \) is a metric.

7. Prove the continuity of the algebraic operations on the real line. (Hint: For multiplication, show \( |x_n y_n - x y| \leq |x_n - x| |y_n - y| + |y_n - y| |x_n - x| + |x_n - x| |y_n - y| \); for division, show first that taking reciprocals is continuous.)

8. Prove: If \( X \) is a topological space and \( f, g : X \to \mathbb{R} \) are continuous then \( f + g \), \( f - g \) and \( f \cdot g \) are continuous. If \( g(x) \neq 0 \) for all \( x \) then \( f / g \) is continuous.

9. In \( \mathbb{R}^n \) (and metric spaces, in general), \( x_n \to x \) means that given \( \varepsilon > 0 \) there is a finite integer \( N \) such that \( d(x_n, x) < \varepsilon \) for all \( n > N \). Show that this agrees with the definition of convergence given for topological spaces.
10. Suppose \( f : X \to Y \) is a map between metric spaces that has the property
\[ d_X(x_1, x_2) = d_Y(f(x_1), f(x_2)) \]
for all points \( x_1 \) and \( x_2 \) in \( X \). Show that \( f \) is an imbedding. It is called an isometric imbedding.

11. Prove: If \( x_n \to x \) and \( y_n \to y \) in \( \mathbb{R} \) then \( x_n + y_n \to x + y \), \( x_n - y_n \to x - y \), \( x_n y_n \to xy \), and (provided \( y_n, y \neq 0 \)) \( x_n / y_n \to x / y \). (Suggestion: Show \( x_n \times y_n \to x \times y \) and use the results of problem 7.)

12. Using the closed set formulation of continuity show that the sets \( \{ (x, y) \mid xy = 1 \} \), \( \{ (x, y) \mid x^2 + y^2 = 1 \} \) and \( \{ (x, y) \mid x^2 + y^2 \leq 1 \} \) are closed in \( \mathbb{R}^2 \).

13. Let \( f_n : \mathbb{R} \to \mathbb{R} \) be defined by
\[ f_n(x) = \frac{1}{n^3[x - (1/n)]^2 + 1} \]
and let \( f(x) = 0 \). Show that \( f_n(x) \to f(x) \) for each \( x \), but \( f_n \) does not converge uniformly to \( f \).

14. Prove the following:
(a) If \( \{s_n\} \) is a bounded sequence of real numbers and \( s_n \leq s_{n+1} \) for each \( n \), then \( \{s_n\} \) converges.

(b) Let \( \{a_n\} \) be a sequence of real numbers. Define \( s_n = \sum_{i=1}^{n} a_i \). If \( s_n \to s \), we say that the infinite series \( \sum_{i=1}^{\infty} a_i \) converges to \( s \). Show that if \( \sum_{i=1}^{\infty} a_i \) converges to \( s \) and \( \sum_{i=1}^{\infty} b_i \) converges to \( t \), then \( \sum_{i=1}^{\infty} ca_i + b_i \) converges to \( cs + t \).

(c) (Comparison test) If \( |a_i| \leq b_i \) for each \( i \) and \( \sum_{i=1}^{\infty} b_i \) converges then \( \sum_{i=1}^{\infty} a_i \) converges. [Hint: First show that \( \sum_{i=1}^{\infty} |a_i| \) converges. Then figure out how to use part (b).]

(d) (Weierstrass M-test) Given \( f_n : X \to \mathbb{R} \), let \( s_n(x) = \sum_{i=1}^{n} f_i(x) \). If \( |f_i(x)| \leq b_i \) for all \( x \) and \( i \) where \( \sum_{i=1}^{\infty} b_i \) converges, then \( s_n(x) \) converges uniformly to a function \( s(x) \).

[Hint: Let \( r_n = \sum_{i=n+1}^{\infty} b_i \) and show that for \( k > n \), \( |s_k(x) - s_n(x)| \leq r_n \).]

15. Let \( f \) be a uniformly continuous real-valued function on a bounded subset \( E \) of the real line. Show that \( f \) is bounded on \( E \). Show that \( f \) need not be bounded if \( E \) is not bounded.
16. Consider the function \( f(x) = \begin{cases} 0 & x \notin \mathbb{Q} \\ 1/n & x = m/n \text{ (here } m \text{ and } n \text{ are relatively prime and } n > 0) \end{cases} \). Prove that \( f \) is continuous at every irrational and discontinuous at every rational.

17. Let \( f : X \to \mathbb{R} \) be a continuous function on a metric space. Show that the zero set \( Z_f = \{ x \mid f(x) = 0 \} \) is closed.

18. If \( A \) is a nonempty subset of a metric space \( X \), define the distance from \( x \) to \( A \) to be \( \delta_A(x) = \text{glb}_{y \in A} d(x, y) \). Prove:
   (a) \( \delta_A(x) = 0 \iff x \in \overline{A} \).
   (b) \( \delta_A \) is uniformly continuous.

19. Let \( A \) and \( B \) be disjoint nonempty closed subsets of a metric space \( X \). Define \( f(x) = \frac{\delta_A(x)}{\delta_A(x) + \delta_B(x)} \). Prove:
   (a) \( f \) is a continuous function whose range lies in \([0,1]\).
   (b) \( f \) is 0 precisely on \( A \) and 1 precisely on \( B \).
   (c) Every closed set in \( X \) is the zero set \( Z_f \) for some continuous function.
   (d) Show that there exists disjoint open sets \( U \) and \( V \) such that \( A \subseteq U \) and \( B \subseteq V \).

20. A subset \( E \) of \( X \) is called \textit{dense} if \( \overline{E} = X \). Suppose \( f, g : X \to Y \) are continuous mappings between metric spaces and that \( E \) is a dense subspace of \( X \). Prove:
   (a) \( f(E) \) is dense in \( f(X) \).
   (b) If \( f(x) = g(x) \) for all \( x \) in \( E \) then \( f(x) = g(x) \) for all \( x \) in \( X \).

21. Suppose \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) is one-to-one and satisfies \( d(x, y) = 1 \) implies that \( d(f(x), f(y)) = 1 \). Show that \( d(x, y) = d(f(x), f(y)) \) for all \( x \) and \( y \).
Additional Homework Problems

1. Let $\mathcal{F}$ be a subset of $Y^X$ such that the set $\{d(f(x), g(x)) \mid x \in X\}$ is bounded $\forall f, g \in \mathcal{F}$.
   Show that $\rho(f, g) = \text{lub} \{d(f(x), g(x)) \mid x \in X\}$ is a metric on $\mathcal{F}$ and that $\bar{\rho} = \min \{\rho(f, g) \mid f, g \in \mathcal{F}\}$.

2. (A) Show that there is a continuous surjection $f : [0,1] \to [0,1]^n$ for any positive integer $n$. [Hint: Consider $f \times f$.]
   (B) Is there a continuous surjection from $[0,1]$ to $\mathbb{R}^2$?

3. A norm on a vector space $X$ is a mapping that assigns to each vector $x$ a real number $\|x\|$ such that:
   (1) $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$,
   (2) $\|x + y\| \leq \|x\| + \|y\|$ and
   (3) $\|cx\| = |c| \|x\|$ for any constant $c$.
   A complete normed vector space is called a Banach space.
   (A) Show that a norm on $X$ can be used to define a metric on $X$.
   (B) Find a metric that is not determined by any norm.
   (C) Consider the space $\ell^\infty$ of all bounded sequences $x = \{x_n\}$ and let $\|x\| = \text{lub} \{|x_n|\}$.
      Show that $\ell^\infty$ is a Banach space.

4. Assume $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, $1 \leq p < \infty$, $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$, and $1/p + 1/q = 1$.
   (A) Show that $a, b \geq 0 \Rightarrow a^{1/p}b^{1/q} \leq a/b + b/q$. [Hint: For $0 < k < 1$ consider the functions $f(t) = t^k$, $g(t) = kt + (1-k)$ when $t \geq 1$.]
   (B) Prove Holder’s inequality: $\sum_{i=1}^{n} |x_i| |y_i| \leq \|x\|_p \|y\|_q$.
   (C) Prove Minkowski’s inequality: $\|x + y\|_p \leq \|x\|_p + \|y\|_p$.
   (D) Show that $x \mapsto \|x\|_p$ is a norm. This shows that the metric from problem 2 is indeed a metric.
   (E) Show that the space $l^n_p$ of all $n$ –tuples $x = (x_1, \ldots, x_n)$ with norm $\|x\|_p$ is a Banach space.
   (F) Compare basic open balls in $\mathbb{R}^2$ for different values of $p$.
   (G) Do Holder’s and Minkowski’s inequalities extend to the case $n = \infty$?

5. A (real) inner product on a vector space $X$ is a map $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R} : (x, y) \mapsto \langle x, y \rangle$ that satisfies:
   (1) $\langle ax + by, z \rangle = a \langle x, z \rangle + b \langle y, z \rangle$,
   (2) $\langle x, y \rangle = \langle y, x \rangle$ and
   (3) $\langle x, x \rangle \geq 0$ with equality occurring iff $x = 0$.
   A complete inner product space is called a Hilbert space.
(A) Prove the *Schwarz inequality*: \( | \langle x, y \rangle | \leq \|x\| \|y\| \).

(B) Show that \( \|x\| = \sqrt{\langle x, x \rangle} \) defines a norm.

(C) Prove the *Parallelogram law*: \( \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \).

(D) Show that the Parallelogram law is not true in \( l_1^2 \). What can you conclude?

(E) Show that \( l_2 = \{ x = \{ x_i \} | (\sum_{i} x_i^2)^{1/2} < \infty \} \) is a Hilbert space.

6. Let \( X \) and \( Y \) be metric spaces. Suppose \( A \) is a subset of \( X \) and \( Y \) is complete. Show that if \( f : A \rightarrow Y \) is uniformly continuous then it can be extended uniquely to a uniformly continuous function \( g : \overline{A} \rightarrow Y \).