A Lower Bound of the Strongly Unique Minimal Projection Constant of $l^n_\infty$, $n \geq 3$.

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Abstract

In this paper we give a lower bound for the strongly unique minimal projection (with norm one) constant (SUP-constant) onto some $(n - k)$-dimensional subspaces of $l^n_\infty$ ($n \geq 3, 1 \leq k \leq n-1$). By Proposition 1 of this paper, each $k$-dimensional Banach space with polytope unit ball with $m$ $(k - 1)$-faces is isometrically isomorphic to a subspace of $l^{k+1-m}_\infty$. As such the aforementioned estimation can be applied to spaces other than $l^n_\infty$. We also include a conjecture about the exact calculations of SUP-constants in particular settings.

1 Introduction

The problem considered in this paper may be treated as a development of the results initiated by G. Lewicki in [4]. If $X$ is a closed linear subspace in Banach space $Y$ then a projection of $Y$ onto $X$ is a bounded linear map $P : Y \to X$ such that $Px = x$ for all $x \in X$. Denote by $\mathcal{P}(Y, X)$ the set of all projections of $Y$ onto $X$.

A projection $P_0$ is called minimal if

$$
\|P_0\| = \lambda(Y, X) = \inf\{\|P\| \mid P \in \mathcal{P}(Y, X)\}.
$$

(1)
A projection \( \pi_0 \in \mathcal{P}(Y, X) \) is called the strongly unique minimal projection (or SUM-projection) if there exists a constant \( s \in (0, 1] \) such that the inequality
\[
\|\pi_0\| + s\|\pi - \pi_0\| \leq \|\pi\|
\]
holds for each \( \pi \in \mathcal{P}(Y, X) \).

It is easy to prove that the SUM-projection \( \pi_0 \) is the unique minimal projection in \( \mathcal{P}(Y, X) \). The largest possible constant for which the inequality in (2) holds is called the strongly unique projection constant (or SUP-constant).

**REMARK 1.** It is known (see for example [1]) that if \( Y = l^n_\infty \) and \( X \subset Y \) is of dimension \( n - 1 \) (\( n \geq 3 \)) with \( X = f^{-1}(0) \) where
\[
f = (f_1, \ldots, f_n) \in Y^*, \ (\|f\|_1 = \max\{|f_1|, \ldots, |f_n|\})
\]
and
\[
0 < f_1 < f_2 < \cdots < f_{n-1} < 1/2, \ f_n \geq 1/2
\]
then the minimal projection \( \pi_0 \) from \( l^n_\infty \) onto \( X \) has norm one and is unique. Moreover, in this case, \( \pi_0 \) is the SUM-projection and the SUP-constant, \( s_0 = s_0(\pi_0) \) is equal to \( 1 - 2f_{n-1} \) ([3], Theorem 2.3.1). Note from [5] that if a minimal projection \( \pi^{00} \) from \( l^n_\infty \) onto \( f^{-1}(0) \) has norm \( u > 1 \) then \( \pi^{00} \) is the SUM-projection and the SUP-constant is equal to
\[
u f_1 \frac{1 - 2f_1}{1 - 2f_1 - uf_1}
\]
where \( f = (f_1, \ldots, f_n) \).

In this paper we consider subspaces \( X = X_{n-k} \subset l^n_\infty, \ 1 \leq k \leq n - 1, \ n \geq 3, \) such that \( \dim X = n - k \). Note that this consideration is quite general due to the following proposition.

**PROPOSITION 1.** Let \( B \) be an \( n \)-dimensional Banach space with unit ball \( U \). Let \( U \) be a polytope with \( m \) \((n-1)\)-dimensional faces. Then \( B \) is isometrically isomorphic to an \( n \)-dimensional subspace of \( l_\infty^{n + m - 1} \)

**Proof.** This follows immediately from Theorem 1 of [2] □

Since we are interested in situations for which the minimal projection onto \( X_{n-k} \) is unique, we may assume without loss generality that (see [10])
\[
X = \cap_{p=1}^k (f^{(p)})^{-1}(0)
\]

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where the hyperplanes \( \{(f^{(p)})^{-1}(0)\}_{p=1}^k \) are given by the linearly independent functionals \( \{f^{(p)}\}_{p=1}^k \in \ell^\infty_\infty^* \) such that, for \( p = 1, \ldots, k \), we have

\[
\|f^{(p)}\|_1 = 1, \quad f^{(p)} = (f^{(p)}_1, \ldots, f^{(p)}_n) \tag{6}
\]

\[
0 < f^{(p)}_1 < f^{(p)}_2 < \cdots < f^{(p)}_{n-k} < \frac{1}{2}, \tag{7}
\]

\[
f^{(1)}_{n-k+1} \geq \frac{1}{2}, \quad f^{(2)}_{n-k+2} \geq \frac{1}{2}, \ldots, f^{(k)}_n \geq \frac{1}{2}. \tag{8}
\]

\[
f^{(p)}_i = 0 \text{ if } p + i \neq n, \quad i = n - k + 1, \ldots, n. \tag{9}
\]

Moreover, if conditions (6) - (9) hold then the unique minimal projection from \( \ell^\infty_\infty \) onto \( X_{n-k} \) has norm one (see [1], Thm. 1; [3], Lemma 2.4.1 and [10], Chp. 2). We will need following two lemmas.

**LEMMA 1** (see [11], pg 89). Let \( Y = \ell^\infty_\infty \) and let \( X = X_{n-k} \) be a subspace of \( Y \) defined by (6) - (9). Then for each projection \( \pi \in \mathcal{P}(Y, X) \) there exists \( k \) elements \( y^{(p)} = (y^{(p)}_1, \ldots, y^{(p)}_k) \in \ell^\infty_\infty, \ p = 1, \ldots, k, \) such that

\[
\pi(x) = x - \sum_{p=1}^k f^{(p)}(x)y^{(p)} \tag{10}
\]

for each \( x \in \ell^\infty_\infty \) and

\[
f^{(q)}(y^{(p)}) = \sum_{i=1}^n y^{(p)}_i f^{(q)}_i = \delta_{qp}, \tag{11}
\]

\( p, q = 1, \ldots, k. \)

**LEMMA 2** (see Introduction of [6]). Let \( Y = \ell^\infty_\infty \) and \( X = X_{n-k} \subset Y \) be defined by (6) - (9). Let \( \pi, \pi_0 \in \mathcal{P}(Y, X) \) and \( y^{(p)}, y^{(p)}(0) \) be elements of \( \ell^\infty_\infty \) which satisfy (10). Then

\[
\|\pi\| = \max_{1 \leq i \leq n} T_i \tag{12}
\]

where

\[
T_i = \sum_{j=1}^n |\delta_{ij} - \sum_{p=1}^k y^{(p)}_i f^{(p)}_j|. \tag{13}
\]
The norm of operator \( \| \pi - \pi_0 \| \) is equal to
\[
\| \pi - \pi_0 \| = \max_{1 \leq i \leq n} B_i
\]
where
\[
B_i = \sum_{j=1}^{n} \left| \sum_{p=1}^{k} (y_i^{(p)} - y_i^{(p)(0)}) f_j^{(p)} \right|, i = 1, \ldots, n.
\]

The main result of this paper is the following.

**THEOREM 1.** Let \( Y = l_\infty^n \) \((n \geq 3)\) and \( X = X_{n-k} \subset Y \) be a subspace of dimension \( n - k \) given by
\[
X_{n-k} = \bigcap_{p=1}^{k} (f^{(p)})^{-1}(0)
\]
where \( \{f^{(p)}\}_{p=1}^{k} \) satisfies (6) - (9). Let \( \pi_0 \) be the minimal projection from \( Y \) onto \( X \). Then \( \pi_0 \) is the SUM-projection with norm one and for the SUP-constant \( s_0 = s(\pi_0) \) we have the inequality
\[
\min \left\{ \frac{f_{n-k+1}^{(1)} - f_{n-k}^{(1)}}{f_{n-k+1}^{(1)} + f_{n-k}^{(1)}}, \frac{f_{n-k+1}^{(2)} - f_{n-k}^{(2)}}{f_{n-k+1}^{(2)} + f_{n-k}^{(2)}}, \ldots, \frac{f_{n-k}^{(k)} - f_{n-k}^{(k)}}{f_{n-k}^{(k)} + f_{n-k}^{(k)}} \right\} \leq s_0 < 1.
\]

**REMARK 2** This result extends the results of O.M. Martynov ([7] and [8]) regarding two and three dimensional subspaces of \( l_\infty^4 \) and \( l_\infty^6 \) respectively. (see Remark 3 below)

**Proof.** We divide the proof into five parts (subsections). First we define the projection \( \pi_0 : l_\infty^n \to X_{n-k} \) which will be the SUM-projection with \( \| \pi_0 \| = 1 \). In the second section we use the fact that the norm of each projection \( \pi \in \mathcal{P}(l_\infty^n, X_{n-k}) \) can be found via (12) and (13) to obtain, for each \( j \in \{n-k, n-k+1, \ldots, n\} \), the inequality
\[
T_j \leq \max\{T_1, T_2, \ldots, T_{n-k-1}\}.
\]
In the third section we determine that for each \( j \in \{n-k, n-k+1, \ldots, n\} \) we have
\[
B_j \leq \max\{B_1, B_2, \ldots, B_{n-k-1}\},
\]
where \( B_i \) \((i = 1, \ldots, n)\) is defined by (14). For the proof of the lower bound of the SUP-constant \( s_0 = s(\pi_0) \) in \((0,1]\) which satisfies (2), it is sufficient to
prove the existence of constants \( s^{(i)} \in (0, 1], i = 1, \ldots, n - k \), for which the inequality
\[
1 + s^{(i)} B_i \leq T_i
\]
holds. This is handled in the fourth section. In the process of finding \( s^{(i)} \) satisfying (19) we establish a lower bound of \( s^{(i)} \):
\[
s_0^{(i)} = \min \left\{ \frac{f^{(1)}_{n-k+1} - f^{(1)}_i}{f^{(1)}_{n-k+1} + f^{(1)}_i}, \frac{f^{(2)}_{n-k+2} - f^{(2)}_i}{f^{(2)}_{n-k+2} + f^{(2)}_i}, \ldots, \frac{f^{(k)}_n - f^{(k)}_i}{f^{(k)}_n + f^{(k)}_i} \right\}, i = 1, \ldots, n - k.
\]

The fifth (and final section) we put
\[
\hat{s} = \min\{s_0^{(1)}, \ldots, s_0^{(n-k)}\}
\]
and establish that the SUP-constant \( s_0 \) satisfies (16); i.e., \( \hat{s} \leq s_0 < 1 \). We now begin the proofs.

1.1 Part 1

We put
\[
y^{(p)0} = (0_1, 0_2, \ldots, 0_{n-k+p-1}, 1/f^{(p)}_{n-k+p}, 0, \ldots, 0) \in l^n_{\infty}, \ p = 1, \ldots, k.
\]

From (9) and (13) we have
\[
T_1 = |1 - y^{(1)}_1 f^{(1)}_1 - \cdots - y^{(k)}_1 f^{(k)}_1| + \cdots + |y^{(1)}_1 f^{(1)}_{n-k} + \cdots + y^{(k)}_1 f^{(k)}_{n-k}|
\]
\[
+ |y^{(1)}_1 f^{(1)}_{n-k+1} + \cdots + y^{(k)}_1 f^{(k)}_{n-k+1}|
\]
\[\vdots\]
\[
T_{n-k} = |y^{(1)}_1 f^{(1)}_{n-k} f^{(1)}_1 + \cdots + y^{(k)}_1 f^{(k)}_1| + \cdots + |1 - y^{(1)}_1 f^{(1)}_{n-k} - \cdots - y^{(k)}_1 f^{(k)}_{n-k}|
\]
\[
+ |y^{(1)}_1 f^{(1)}_{n-k+1} + \cdots + y^{(k)}_1 f^{(k)}_{n-k+1}|
\]
\[
T_{n-k+r} = |y^{(1)}_1 f^{(1)}_{n-k+r} f^{(1)}_1 + \cdots + y^{(k)}_1 f^{(k)}_1| + \cdots + y^{(1)}_1 f^{(1)}_{n-k+r} f^{(1)}_{n-k} + \cdots + y^{(k)}_1 f^{(k)}_{n-k+r} f^{(k)}_{n-k}|
\]
\[
+ |y^{(1)}_1 f^{(1)}_{n-k+r} f^{(1)}_{n-k+1} + \cdots + y^{(k)}_1 f^{(k)}_{n-k+r} f^{(k)}_{n-k+1}|, \ r = 1, \ldots, k
\]

Then by (22) we get \( T^{(0)}_1 = T^{(0)}_2 = \cdots = T^{(0)}_{n-k} = 1 \) and
\[
T^{(0)}_{n-k+1} = \frac{f^{(1)}_{n-k+1}}{f^{(1)}_{n-k+1}} + \frac{f^{(1)}_{n-k+1}}{f^{(1)}_{n-k+1}} + \cdots + \frac{f^{(1)}_{n-k}}{f^{(1)}_{n-k+1}} = 1 - \frac{f^{(1)}_{n-k+1}}{f^{(1)}_{n-k+1}}
\]
\[ T_{n-k+2}^{(0)} = \frac{f_{n-k+2}^{(2)}}{f_{n-k+2}^{(2)}} + \frac{f_{n-k}^{(2)}}{f_{n-k+2}^{(2)}} + \cdots + \frac{f_{n-k}^{(2)}}{f_{n-k+2}^{(2)}} = 1 - f_{n-k+2}^{(2)} \]

\[ \vdots \]

\[ t_n^{(0)} = \frac{f_1^{(k)}}{f_n^{(k)}} + \frac{f_2^{(k)}}{f_n^{(k)}} + \cdots + \frac{f_{n-k}^{(k)}}{f_n^{(k)}} = 1 - f_n^{(k)}. \]

Taking into account \( f_{n-k+i}^{(i)} \geq 1/2, i = 1, \ldots, k \), we find

\[ 1 - f_{n-k+p}^{(p)} \leq 1 \]

and therefore \( T_{n-k+p}^{(0)} \leq 1, p = 1, \ldots, k \). Thus by (12) we have \( \|\pi_0\| = 1 \). By conditions (6) - (9) the projection \( \pi_0 \) is the unique minimal projection onto \( X_{n-k} \) (see [10]).

### 1.2 Part 2

We now evaluate from (11) \( y_j^{(i)} \) \( (i = 1, \ldots, k; j = n - k + 1, \ldots, n) \) with the help of \( y_r^{(m)} \), where \( m = 1, \ldots, k \) and \( r = 1, \ldots, n - k \):

\[
y_j^{(i)} = \begin{cases} 
\frac{1}{f_j^{(i)}} (1 - y_1^{(i)} f_1^{(i)} - \cdots - y_{n-k}^{(i)} f_{n-k}^{(i)}) & \text{if } j - i = n - k, \\
\frac{1}{f_j^{(i-(n-k))}} (y_1^{(i)} f_1^{(j-(n-k))} + \cdots + y_{n-k}^{(i)} f_{n-k}^{(j-(n-k))}) & \text{if } j - i \neq n - k.
\end{cases}
\]

(23)

Using the expressions for \( T_i \) \( (i = 1, \ldots, n) \) given in Part 1 and (23) we obtain via the triangle inequality

\[ T_{n-k+r} \leq \frac{f_1^{(r)}}{f_{n-k+r}^{(r)}} T_1 + \frac{f_2^{(r)}}{f_{n-k+r}^{(r)}} T_2 + \cdots + \frac{f_{n-k}^{(r)}}{f_{n-k+r}^{(r)}} T_{n-k}, \quad r = 1, \ldots, k. \]

(24)

By virtue of (6) - (9) we have

\[ \frac{f_1^{(r)}}{f_{n-k+r}^{(r)}} + \cdots + \frac{f_{n-k}^{(r)}}{f_{n-k+r}^{(r)}} = 1 - \frac{f_{n-k+r}^{(r)}}{f_{n-k+r}^{(r)}} \leq 1 \]

and therefore

\[ T_{n-k+r} \leq \max\{T_1, \ldots, T_{n-k+r}\}, \quad r = 1, \ldots, k. \]

Thus

\[ \|\pi\| = \max\{T_1, \ldots, T_{n-k}\}. \]

(25)
1.3 Part 3

We now consider the quantities $B_1, \ldots, B_{n-k}, B_{n-k+r}$, $r = 1, \ldots, k$. Note first that

$$B_1 = |y_1^{(1)} f_1^{(1)} + y_1^{(2)} f_1^{(2)} + \cdots + y_1^{(k)} f_1^{(k)}| + \cdots$$

$$+ |y_1^{(1)} f_{n-k}^{(1)} + y_1^{(2)} f_{n-k}^{(2)} + \cdots + y_1^{(k)} f_{n-k}^{(k)}|$$

$$+ |y_1^{(1)} f_{n-k+1}^{(1)} + y_1^{(2)} f_{n-k+1}^{(2)} + \cdots + y_1^{(k)} f_{n-k}^{(k)}|,$$

$$\vdots$$

$$B_{n-k} = |y_{n-k}^{(1)} f_1^{(1)} + \cdots + y_{n-k}^{(k)} f_1^{(k)}| + \cdots$$

$$+ |y_{n-k}^{(1)} f_{n-k}^{(1)} + \cdots + y_{n-k}^{(k)} f_{n-k}^{(k)}|$$

$$+ |y_{n-k}^{(1)} f_{n-k+1}^{(1)} + \cdots + y_{n-k}^{(k)} f_{n-k}^{(k)}|.$$ 

Therefore, for $r = 1, \ldots, k$, we have

$$B_{n-k+r} = |y_{n-k+r}^{(1)} f_1^{(1)} + \cdots + (y_{n-k+r}^{(r)} - \frac{1}{f_{n-k+r}^{(r)}} f_{n-k+r}^{(r)}) f_1^{(r)} + \cdots + y_{n-k+r}^{(k)} f_1^{(k)}| + \cdots$$

$$+ |y_{n-k+r}^{(1)} f_{n-k}^{(1)} + \cdots + (y_{n-k+r}^{(r)} - \frac{1}{f_{n-k+r}^{(r)}} f_{n-k+r}^{(r)}) f_{n-k}^{(r)} + \cdots + y_{n-k+r}^{(k)} f_{n-k}^{(k)}|$$

$$+ |y_{n-k+r}^{(1)} f_{n-k+1}^{(1)} + \cdots + (y_{n-k+r}^{(r)} - \frac{1}{f_{n-k+r}^{(r)}} f_{n-k+r}^{(r)}) f_{n-k+1}^{(r)} + \cdots + y_{n-k+r}^{(k)} f_{n-k+1}^{(k)}|$$

$$+ \cdots + y_{n-k+r}^{(k)} f_n^{(k)}.$$ 

Using (23) and the triangle inequality as in (24) we obtain

$$B_{n-k+r} \leq \frac{f_1^{(r)}}{f_{n-k+r}^{(r)}} B_1 + \frac{f_2^{(r)}}{f_{n-k+r}^{(r)}} T_2 + \cdots + \frac{f_{n-k}^{(r)}}{f_{n-k+r}^{(r)}} B_{n-k}, \quad r = 1, \ldots, k \tag{26}$$

and thus

$$B_{n-k+r} \leq \max\{B_1, \ldots, B_{n-k}\} \tag{27}$$

for $r = 1, \ldots, k$.

1.4 Part 4

By (26) and (27) we can write the inequality (2) in the following form:

$$1 + s \max\{B_1, \ldots, B_{n-k}\} \leq \max\{T_1, \ldots, T_{n-k}\} \tag{28}$$
If we show that, for each $i \in \{1, \ldots, n - k\}$, there exists $s \in (0, 1]$ such that
\[ 1 + sB_i \leq T_i \] (29)
then this will establish (28).

Let $i = 1$. The existence of $s \in (0, 1]$ such that
\[ 1 + sB_1 \leq T_1 \] (30)
is equivalent to the existence of $s \in (0, 1]$ such that
\[
1 + s\{|y_1^{(1)}f_1^{(1)}| + \cdots + y_1^{(k)}f_1^{(k)}| + \cdots + |y_1^{(1)}f_{n-k} + \cdots + y_1^{(k)}f_{n-k}|
+ |y_1^{(1)}|\tilde{f}_{n-k+1} + \cdots + |y_1^{(k)}|\tilde{f}_{n-1}\} \]

is less than or equal to
\[
|1 - y_1^{(1)}f_1^{(1)} - \cdots - y_1^{(k)}f_1^{(k)}| + |y_1^{(1)}|\tilde{f}_{n-k+1} + \cdots + |y_1^{(k)}|\tilde{f}_{n-1}. \]

To demonstrate this inequality, it is sufficient to verify that
\[ 1 + s\{|y_1^{(1)}f_1^{(1)}| + \cdots + y_1^{(k)}f_1^{(k)}| + |y_1^{(1)}|\tilde{f}_{n-k+1} + \cdots + |y_1^{(k)}|\tilde{f}_{n-1}\} \] (31)
is less than or equal to
\[
|1 - y_1^{(1)}f_1^{(1)} - \cdots - y_1^{(k)}f_1^{(k)}| + |y_1^{(1)}|\tilde{f}_{n-k+1} + \cdots + |y_1^{(k)}|\tilde{f}_{n-1}. \] (32)

Note that (31) is less than or equal to
\[ 1 + s\{|y_1^{(1)}|\tilde{f}_{1}^{(1)} + \tilde{f}_{n-k+1}^{(1)}| + |y_1^{(2)}|\tilde{f}_{1}^{(2)} + \tilde{f}_{n-k+2}^{(2)} + \cdots + |y_1^{(k)}|\tilde{f}_{1}^{(k)} + \tilde{f}_{n}^{(k)}\} \]
and (32) is greater than or equal to
\[ 1 + |y_1^{(1)}|\tilde{f}_{n-k+1}^{(1)} - \tilde{f}_{1}^{(1)}| + |y_1^{(2)}|\tilde{f}_{n-k+2}^{(2)} - \tilde{f}_{1}^{(2)} + \cdots + |y_1^{(k)}|\tilde{f}_{n}^{(k)} - \tilde{f}_{1}^{(k)}. \]

Therefore for the existence of $s \in (0, 1]$ such that (31) is less than or equal to (32), it is sufficient to prove that for some $s \in (0, 1]$ we have
\[ 1 + |y_1^{(1)}|\tilde{f}_{n-k+1}^{(1)} - \tilde{f}_{1}^{(1)}| + |y_1^{(2)}|\tilde{f}_{n-k+2}^{(2)} - \tilde{f}_{1}^{(2)} + \cdots + |y_1^{(k)}|\tilde{f}_{n}^{(k)} - \tilde{f}_{1}^{(k)} \] (33)
greater than or equal to
\[
1 + s\{y_1^{(1)}|f_1^{(1)} + f_{n-k+1}^{(1)}| + \cdots + y_1^{(k)}|f_1^{(k)} + f_n^{(k)}|\}. \tag{34}
\]
Note that (33) greater than or equal to (34) is equivalent to existence of \(s \in (0, 1]\) such that
\[
s \leq \frac{|y_1^{(1)}|(f_{n-k+1} - f_1^{(1)}) + \cdots + |y_1^{(k)}|(f_n - f_1^{(k)})}{|y_1^{(1)}|(f_{n-k+1}^{(1)} + f_1^{(1)}) + \cdots + |y_1^{(k)}|(f_n^{(k)} + f_1^{(k)})}. \tag{35}
\]
Now denote by \(S\) the set of values of the right-hand side of (35); for example,
\[
\frac{f_{n-k+1}^{(1)} - f_1^{(1)}}{f_{n-k+1}^{(1)} + f_1^{(1)}} \in S
\]
in the case
\[
y_1^{(1)} \neq 0, y_1^{(2)} = \ldots y_1^{(k)} = 0.
\]
Let
\[
s_1 = \min \left\{ \frac{f_{n-k+1}^{(1)} - f_1^{(1)}}{f_{n-k+1}^{(1)} + f_1^{(1)}}, \ldots, \frac{f_{n-k+1}^{(k)} - f_1^{(k)}}{f_{n-k+1}^{(k)} + f_1^{(k)}} \right\}. \tag{36}
\]
Then it is easy to see if \(s \in (0, s_1]\) then (35) holds and, consequently, (30) also holds.

1.5 Part 5

In an analogous way, for \(j = 2, \ldots, n - k\), the inequality
\[
1 + sB_j \leq T_j \tag{37}
\]
is correct for each \(s \in (0, s_j]\), where
\[
s_j = \min \left\{ \frac{f_{n-k+1}^{(1)} - f_j^{(1)}}{f_{n-k+1}^{(1)} + f_j^{(1)}}, \ldots, \frac{f_{n-k+1}^{(k)} - f_j^{(k)}}{f_{n-k+1}^{(k)} + f_j^{(k)}} \right\}.
\]
By (7), we have for each \(r \in \{1, \ldots, k\}\) the inequality
\[
\frac{f_{n-k+1}^{(r)} - f_n^{(r)}}{f_{n-k+1}^{(r)} + f_n^{(r)}} < \frac{f_{n-k+1}^{(r)} - f_{n-k-1}^{(r)}}{f_{n-k+1}^{(r)} + f_{n-k-1}^{(r)}} < \cdots < \frac{f_{n-k+1}^{(r)} - f_1^{(r)}}{f_{n-k+1}^{(r)} + f_1^{(r)}}.
\]
Therefore, for each $s \in (0, \hat{s}]$ we have

$$1 + s\|\pi - \pi_0\| \leq \|\pi\|$$

where

$$\hat{s} = \min \left\{ \frac{f_{n-k+1}^{(1)} - f_{n-k}^{(1)}}{f_{n-k+1}^{(1)} + f_{n-k}^{(1)}}, \ldots, \frac{f_{n-k+1}^{(k)} - f_{n-k}^{(k)}}{f_{n-k+1}^{(k)} + f_{n-k}^{(k)}} \right\} \leq \min\{s_1, \ldots, s_{n-k}\}. \quad (38)$$

2 Concluding Remarks

REMARK 3. In general

$$\hat{s} = \min \left\{ \frac{f_{n-k+1}^{(1)} - f_{n-k}^{(1)}}{f_{n-k+1}^{(1)} + f_{n-k}^{(1)}}, \ldots, \frac{f_{n-k+1}^{(k)} - f_{n-k}^{(k)}}{f_{n-k+1}^{(k)} + f_{n-k}^{(k)}} \right\} \leq \min\{s_1, \ldots, s_{n-k}\}.$$

is the not equal the SUP-constant; indeed in the case $k = 1, n \geq 3$ we have

$$\hat{s} = \frac{f_n^{(1)} - f_{n-1}^{(1)}}{f_n^{(1)} + f_{n-1}^{(1)}} < \frac{f_n^{(1)} - f_{n-1}^{(1)} + (f_n^{(1)} + \cdots + f_1^{(1)})}{f_n^{(1)} + f_{n-1}^{(1)} + (f_n^{(1)} + \cdots + f_1^{(1)})} = 1 - 2f_n^{(1)}$$

and, by [3](Thm 2.3.1), $1 - 2f_n^{(1)}$ is the SUP-constant.

COROLLARY 1. If $k = n - 1, n \geq 3$, then under the hypotheses of Theorem 1 we have

$$\hat{s} = \min\{f_2^{(2)} - f_1^{(1)}, \ldots, f_n^{(n-1)} - f_1^{(n-1)}\}.$$

Proof. This follows from (38) since, in this case,

$$f_2^{(1)} + f_1^{(1)} = \cdots = f_n^{(n-1)} + f_1^{(n-1)} = 1$$

by (6) and (9).

REMARK 4. In [6] a SUP-constant is established for the minimal projections onto some two-dimensional subspaces of $l_4^\infty$. The subspaces are defined as the intersection of two hyperplanes: $H_1 = f_1^{-1}(0)$ where $f_1 = (1, s, r, 0) \in$ 10
Theorem 4.1: Let \( \ell_1^4 \) with \( s > 0, \ r > 0 \) and \( H_2 = f_2^{-1}(0) \) where \( f_1 = (0, 0, 0, 1) \). In this case the constant \( k \) of strong unicity is

\[
k = \frac{r(1 - s + r)(s + r - 1)}{(1 - s)^2 + r(1 + s)(1 + s + r)}, \quad \text{if } s + r - 1 \geq 0, \ r \leq s \leq 1
\] (39)

and

\[
k = \frac{1 + r - s}{1 + r + s}, \quad \text{if } 1 - s - r \geq 0, \ r \leq s.
\] (40)

It is interesting to note that if we consider only \( \ell_\infty^3 \) and \( \hat{H}_1 = \hat{f}_1^{-1}(0) \), where \( \hat{f}_1 = (r, s, 1) \), then the SUP-constants are the same as in (39) and (40). To verify this, let

\[
\hat{f}_1 = \frac{\hat{f}_1}{\|\hat{f}_1\|} = \left( \frac{r}{1 + r + s}, \frac{s}{1 + r + s}, \frac{1}{1 + r + s} \right).
\]

In this case \( \hat{H}_1 = \hat{f}_1^{-1}(0) \) and we consider the following two cases. If \( 1/2 \leq \frac{1}{1 + s + r} (\iff 1 - s - r \geq 0) \) then the minimal projection from \( \ell_\infty^3 \) onto \( \hat{H}_1 \) has norm one and is unique (see [1]). Moreover by the result of G. Lewicki ([3]), we have

\[
k = 1 - 2\frac{s}{1 + r + s} = \frac{1 + r - s}{1 + r + s}.
\]

If \( \frac{1}{1 + s + r} < 1/2 (\iff s + r - 1 \geq 0) \) then the minimal projection from from \( \ell_\infty^3 \) onto \( \hat{H}_1 \) has norm \( 1 + u \) and is unique (see [5] and [9]) where

\[
u = \left( \sum_{i=1}^{3} \frac{f_i}{1 - 2f_i} \right)^{-1}
\]

with \( f = (f_1, f_2, f_3), \ 0 < f_1 \leq f_2 \leq f_3 < 1/2 \) and \( \|f\|_1 = 1 \). In our situation we have

\[
f_1 = \frac{r}{1 + s + r}, \ f_2 = \frac{s}{1 + s + r}, \ f_3 = \frac{1}{1 + s + r}
\]

and thus

\[
u = \left( \frac{r}{1 + s + r} + \frac{s}{1 + s + r} + \frac{1}{1 + s + r} \right)^{-1}.
\]

Moreover, by (4), we find

\[
k_0 = uf_1 \frac{1 - 2f_1}{1 - 2f_1 - uf_1} = \frac{r(1 - s + r)(s + r - 1)}{(1 - s)^2 + r(1 + s)(1 + s + r)}.
\]
As a consequence of the above Remark and the symmetry of $l_\infty^n$ spaces we make the following conjecture.

**CONJECTURE 1**. Let $f^{(0)}, \ldots, f^{(n-k)} \in (l_\infty^n)^*$, where

\[
\begin{align*}
  f^{(0)} &= (f_1, \ldots, f_k, 0_{k+1}, \ldots, 0_n), \\
  f^{(1)} &= (0_1, \ldots, 0_k, 1, 0_{k+2}, \ldots, 0_n), \\
  f^{(2)} &= (0_1, \ldots, 0_{k+1}, 1, 0_{k+3}, \ldots, 0_n), \\
  & \vdots \\
  f^{(n-k)} &= (0_1, \ldots, 0_{n-1}, 1)
\end{align*}
\]

where $\sum_{i=1}^k f_i^{(0)} = 1$ and $0 < f_1 \leq f_2 \leq \cdots \leq f_{k-1} < f_k$. Let $\hat{f} = (f_1, \ldots, f_k) \in (l_\infty^k)^*$ and let $\pi_{\hat{f}}$ be a unique minimal projection onto $(\hat{f})^{-1}(0)$ from $l_\infty^k$. Let

\[ H = \bigcap_{p=0}^{n-k} (f^{(p)})^{-1}(0) \]

and $\pi_H$ be a unique minimal projection onto $H$ from $l_\infty^n$. Then the SUP-constant $k(\pi_H)$ is equal to SUP-constant $k(\pi_{\hat{f}})$.

**References**


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