Non-existence of Monotonically Complemented Subspaces of $C[a, b]$ and Other Negative Results

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Abstract

A subspace $V$ of a Banach space $X$ is said to be complemented if there exists a (bounded) projection mapping $X$ onto $V$. Obviously all subspaces of finite-dimension are complemented. The goal of this note is to show that there are (relatively) few monotonically complemented subspaces of finite-dimension in $X = (C[a, b], ||\cdot||_\infty)$; that is, finite-dimensional subspaces $V \subset X$ for which there exists a projection $P : X \to V$ such that $Pf$ is monotone-increasing whenever $f$ is. We obtain several corollaries from this consideration, including a result describing the difficulty of preserving $n$-convexity via a projection.

1 Introduction and Preliminaries

By a cone $S$ of (real) Banach space $X$ we mean a convex subset of $X$ which is closed under nonnegative scalar multiplication. Every cone $S \subset X$ contains the origin and a pointed cone contains no lines through the origin. Let $L(X)$ denote the set of linear operators on $X$. For a given cone $S$ and operator $Q \in L(X)$, a natural question arises: does $Q$ leave $S$ invariant? There are numerous settings in which knowing the answer to this has important consequences and connections. For example, under the right conditions on $S$, $X$ becomes a Banach lattice and $Q$ such that $QS \subset S$ becomes a positive operator (see [5] for an overview). Existence of positive operators (or more
precisely positive extensions) is employed, for example, in the Korovkin’s classical theorem (described in [2]) and in its many generalizations (see for example [3]).

Outside of the Banach lattice realm, \( Q \in \mathcal{L}(X) \) such that \( QS \subset S \) is often called a cone-preserving map (see [6] for an extensive description). Borrowing this terminology, for given cone \( S \) let us denote the set of all cone-preserving operators by \( \mathcal{L}_S(X) \) (or by \( \mathcal{L}_S \) when there no chance of ambiguity). Not surprisingly, the determination of whether or not a given \( Q \in \mathcal{L} \) belongs to \( \mathcal{L}_S \) can be quite difficult. Indeed, one finds in the literature that existence of 'cone-preserving' operators is frequently considered only in the case in which \( X \) is finite-dimensional. The fact that membership in \( \mathcal{L}_S \) is very “sensitive” to \( X, S \) and \( Q \) certainly contributes to the difficulty. For example, there is no finite-rank operator in \( \mathcal{L}_S(X) \) which fixes \( V \), where \( X = (C[0, 1], \| \cdot \|_\infty) \), \( S \) is the cone of nonnegative elements from \( X \) and \( V = \Pi_2 = [1, x, x^2] \), the space of second-degree algebraic polynomials (spanned by \( \{1, x, x^2\} \)). However, if instead we require fixing \( \Pi_1 \) and \( x^2 \mapsto (x + x^2)/2 \) - i.e., nearly fixing \( V \) - then such an operator does belong to \( \mathcal{L}_S(X) \). Or instead, consider the fact that, while there exists no projection from \( X \) onto \( V = \Pi_2 \) preserving monotonicity , it is possible to project \( X_1 \) onto \( V \) and leave the cone of monotone functions (of \( X_1 \)) invariant, where \( X_1 \) is the (Banach) space of \( C^1 \) functions on \([0, 1]\) normed by \( \|f\|_{X_1} := \max\{\|f\|_\infty, \|f'\|_\infty\} \).

There has been some progress made in characterizing (in a useful way) membership in \( \mathcal{L}_S(X) \) in particular settings within the shape-preserving projection realm. In this setting, we fix \( X \) and finite-dimensional subspace \( V \subset X \) and say that \( P \in \mathcal{L}(X) \) is a projection (onto \( V \)) if \( P : X \to V \) such that \( P|_V = I \), the identity map. We denote the set of all projections by \( \mathcal{P}(X, V) = \mathcal{P} \). We consider \( \mathcal{P}_S(X, V) \) for various (well-structured) cones \( S \subset X \), where \( \mathcal{P}_S(X, V) = \mathcal{P}_S \) denotes the set of projections from \( \mathcal{P} \) such that \( PS \subset S \). We say \( P \) is shape-preserving (with respect to \( S \)) if \( P \in \mathcal{P}_S \). The 'well-structured' condition is actually a property that the dual cone \( S^* \subset X^* \) of \( S \) must possess, where

\[
S^* := \{u \in X^* \mid u(f) \geq 0 \, \forall f \in S\}.
\]

Specifically, recent papers on existence of shape-preserving operators (see e.g. [4]) have required that \( S^* \) be simplicial; that is, \( S^* \) must be recoverable from its set extreme rays and this set should consist of 'independent' elements (independent in a representing measure (or Choquet) sense). For example,
if $S$ denotes the cone of monotone functions in $X_1$ above, then one can show that $S^*$ is simplicial; however, $S^*$ fails to be simplicial when $S$ is the cone of monotone functions from $X = (C[0,1], \| \cdot \|_\infty)$. This fact, proven below, provides that basis of our investigation.

Following these introductory remarks are two sub-sections which describe some general, and then specific, properties of cones. Section 2 begins with the main theorem of this paper and then several corollaries are proven. The proof of Theorem 1 is contained the subsequent subsection.

1.1 Some general properties of cones

In a (real) topological vector space, a cone $K$ is a convex set, closed under nonnegative scalar multiplication. $K$ is pointed if it contains no lines. For $\phi \in K$, let $[\phi]^+ := \{ \alpha \phi \mid \alpha \geq 0 \}$. We say $[\phi]^+$ is an extreme ray of $K$ if $\phi = \phi_1 + \phi_2$ implies $\phi_1, \phi_2 \in [\phi]^+$ whenever $\phi_1, \phi_2 \in K$. We let $E(K)$ denote the union of all extreme rays of $K$. When $K$ is a closed, pointed cone of finite dimension we always have $K = \text{co}(E(K))$ (this need not be the case when $K$ is infinite dimensional; indeed, we will see that it is possible that $E(K) = \emptyset$ despite $K$ being closed and pointed).

DEFINITION 1 Let $X$ be a Banach space, $V \subset X$ a subspace and let $X^*$ denote the dual space of $X$. Let $S \subset X$ denote a closed cone. We say that $x \in X$ has shape (in the sense of $S$) whenever $x \in S$. If $P \in \mathcal{P} = \mathcal{P}(X,V)$ and $PS \subset S$ then we say $P$ is a shape-preserving projection; we denote the set of all such projections by $\mathcal{P}_S = \mathcal{P}_S(X,V)$. For a given cone $S$, define $S^* = \{ \phi \in X^* \mid \langle x, \phi \rangle \geq 0 \ \forall x \in S \}$. We will refer to $S^*$ as the dual cone of $S$.

NOTE 1 In determining the existence of shape-preserving operators mapping $X$ onto $d$-dimensional subspace $V \subset X$ (and preserving $S$), we may always assume that there a basis for $V$ contained in $S$; i.e., $\dim V \cap S = d$. Indeed, let $k$ denote the size of the largest linearly independent subset of $V \cap S$, where integer $k \in [1, \ldots , d]$. Label as $v_1, \ldots , v_k$ a linearly independent set from this intersection. Choose $v_{k+1}, \ldots , v_d \in V$ so that $v_1, \ldots , v_d$ forms a basis for $V$. With this basis, any operator $P : X \to V$ can be expressed in the form $P = u_1 \otimes v_1 + \cdots + u_n \otimes v_d$ for some choice of $u_i$'s in $X^*$, where $Pf = u_1(f)v_1 + \cdots + u_d(f)v_d$. Using this representation, note that, if $PS \subset S$ then $\sum_{i=k+1}^d u_i(x)v_i = 0$ for every $x \in S$ (since otherwise the definition of
would be violated). That is, \( PS \subset S \) if and only if \( P_1S \subset S \), where \( P_1 = u_1 \otimes v_1 + \cdots + u_k \otimes v_k \). Thus we may as well assume \( k = d \).

The following Lemma indicates that \( S^* \) is in fact "dual" to \( S \).

**Lemma 1** Let \( x \in X \). If \( \langle x, \phi \rangle \geq 0 \) for all \( \phi \in S^* \) then \( x \in S \).

**Proof.** We prove the contrapositive; suppose \( x \in X \) such that \( x \notin S \). Then, since \( S \) is closed and convex, there exists a separating functional \( \phi \in X^* \) and \( \alpha \in \mathbb{R} \) such that \( \phi(x) < \alpha \) and

\[
\phi(s) > \alpha \quad \forall s \in S. 
\]

(1)

Note that we must have \( \alpha < 0 \) because \( 0 \in S \). In fact, for every \( s \in S \) we claim

\[
\phi(s) \geq 0 > \alpha. 
\]

(2)

To check this, suppose there exists \( s_0 \in S \) such that \( \phi(s_0) = \beta < 0 \); this would imply

\[
\phi\left(\frac{\alpha}{\beta}s_0\right) = \alpha 
\]

while \( \frac{\alpha}{\beta}s_0 \in S \). And this is in contradiction to (1). The validity of (2) implies that \( \phi \in S^* \) and this completes the proof. \( \blacksquare \)

**Lemma 2** Let \( P \in \mathcal{P}(X,V) \). Then \( PS \subset S \iff P^*S^* \subset S^* \), where \( P^* : X^* \rightarrow X^* \) denotes the adjoint of \( P \).

**Proof.** The proof is an immediate consequence of the duality equation \( \phi(Px) = P^*\phi(x) \) and Lemma 1. \( \blacksquare \)

**Lemma 3** Let \( X \) denote a Banach space and \( S \subset X \) a pointed cone. Let \( V \subset X \) be a finite-dimensional subspace. If \( \mathcal{P}_S \neq \emptyset \) then the cone \( S^*_V \) is closed.

**Proof.** Let \( d := \dim(V) \). Let \( P \in \mathcal{P}_S \) and let \( v = [v_1, \ldots, v_d]^T \) denote a fixed basis for \( V \). Let \( \overline{P^*S^*} \) denote the closure of \( P^*S^* \) and let \( P^* \phi \in \overline{P^*S^*} \subset P^*X^* \). Choose a sequence \( \{P^*\phi_k\}_{k=1}^{\infty} \subset P^*S^* \) such that \( P^*\phi_k \rightarrow P^*\phi \). Notice, by Lemma 2, \( \{P^*\phi_k\}_{k=1}^{\infty} \subset S^* \). \( S^* \) is weak*-closed and therefore
\( P^* \phi \in S^* \); this implies \( P^* \phi \in P^* S^* \) since \((P^*)^2 = P^* \). Thus \( P^* S^* \) is closed. Note that \( P^* S^* \) is homeomorphic to \((P^* S^*)|_V \) and thus \((P^* S^*)|_V \) is closed. Finally, we claim \((P^* S^*)|_V = S^*|_V \). To verify this, choose \( \phi \in S^*, v \in V \) and consider

\[
P^* \phi(v) = \phi(Pv) = \phi(v),
\]

where the last equality follows from the fact that \( P \) is a projection. But this equation simply says that \( P^* \phi \) and \( \phi \) agree on \( V \), thus establishing the claim. From here we can conclude that \( S^*|_V \) is closed.

\section*{1.2 The cones of interest}

For the remainder of this paper, the Banach space \((C[a,b], \| \cdot \|_\infty) \) will be denoted by \( X \) with dual space denoted by \( X^* \). A function \( f \in X \) is said to be \textit{monotone} (increasing) if \( f(s) \leq f(t) \) whenever \( s < t \). \( f \in X \) is \textit{strictly monotone} if the former inequality is strict whenever the latter is. Let \( S_1 \subset X \) denote the cone of all monotone functions, with dual cone denoted by \( S_1^* \subset X^* \) (rather than the less convenient \((S_1)^* \)).

**Definition 2** Subspace \( V \subset X \) is said to be monotonically complemented if there exists a (bounded) projection \( P \in \mathcal{P}(X,V) \) such that \( PS_1 \subset S_1 \); i.e., if \( \mathcal{P}_{S_1}(X,V) \neq \emptyset \).

This paper will demonstrate a 'lack' of monotonically complemented subspaces. This shortage is, at least in part, due to the unusual structure of \( S_1^* \), which we describe below and exploit in Lemma 6.

**Lemma 4** \( S_1^* \) is a weak* closed, pointed cone possessing no extreme rays.

**Proof.** From the definition of \( S_1^* \) it is immediate that this set is a weak* closed cone; if \( S_1^* \) contained a line, \([u]\), then it would be necessary for \( u \) to vanish against every monotone function in \( X \) and therefore \( u = 0 \). Thus \( S_1^* \) is pointed. We now demonstrate that every element of the cone \( S_1^* \) is a midpoint of line segment belonging to the cone. Let \( \mu \in S_1^* \); then there exists a point \( t \in (a,b) \) such that \( \mu(t) = 0 \). Let \( E_0 := \text{supp}(\mu) \cap [a,t) \) and \( E_1 := \text{supp}(\mu) \cap (t,b] \). For \( i = 0, 1 \) define

\[
\mu_i := \mu|_{E_i} + (-1)^i m \delta_t \quad (3)
\]
where \( m := \mu(E_1) \). Note that \( \mu_i \in X^* \) for each \( i \); in fact, we claim that \( \mu_i \in S_1^* \) for each \( i \). To see this, let \( f(x) \in S \) and define
\[
\hat{f}(x) := \begin{cases} f(x) & \text{if } x \leq t, \\ f(t) & \text{if } x > t. \end{cases}
\]

Then \( \int_a^b f \, d\mu_0 = \int_t^b f \, d\mu + mf(t) = \int_a^b \hat{f} \, d\mu \geq 0 \), where this last inequality follows from the fact that \( \hat{f}(x) \in S \). Thus \( \mu_0 \in S_1^* \) by. A similar argument (using the observation that \( \mu(E_0) = -m \)) shows that \( \mu_1 \in S_1^* \). Finally, notice that for each \( \mu \)-measurable set \( A \) we have
\[
\mu(A) = \mu(A \cap E_0) + \mu(A \cap E_1) = \mu_0(A) + \mu_1(A)
\]
and since \( \mu_0 \) and \( \mu_1 \) do not belong to a common ray of \( S_1^* \) we conclude has \( S_1^* \) has no extreme rays. \qed

**NOTE 2** The approach used in Lemma 4 can be extended to Banach spaces other than \( X = (C[a, b], \| \cdot \|_\infty) \). For example, using the notation of [1], we say \( f \in X \) is \( n \)-convex \((n \geq 2)\) if, for all choices of \( n + 1 \) distinct points \( s_0 < s_1 < \cdots < s_n \) in \( [a, b] \), we have the \( n \)-th divided difference
\[
V_n(f; s_i) \geq 0.
\]
For example, \( n = 2 \) is standard convexity. Denote the cone of all \( n \)-convex functions by \( S_n \). For integer \( k \geq 1 \), let \( C^k[a, b] \) denote the space of \( k \)-continuously differentiable functions and define
\[
X_k := (C^k[a, b], \| \cdot \|_{(k)}) \text{ where } \| f \|_{(k)} := \max_{j=0,\ldots,k} \| f^{(j)} \|_{\infty}.
\]

It is well known that \( S_n \subset X_r \) for \( r = 0, \ldots, n-2 \) (here \( X_0 := X \)); however, consider instead the cone \( T_n := S_n \cap X_{n-1} \) (for example, \( T_2 \) is the cone of smooth \((C^1)\) convex functions). Then \( T_n^* \subset X_{n-1}^* \) has no extreme rays (despite being pointed and weak* closed). The proof uses the same argument as that of Lemma 4 but with (3) replaced by
\[
\mu_i := |E_i| + (-1)^i \left( m_0 \delta_t + \sum_{k=1}^{n-2} m_k \delta_i^{(k)} + m_{n-1} (\delta_s^{(n-2)} - \delta_t^{(n-2)}) \right)
\]
where \( \delta_s \) (and \( \delta_s^{(0)} \)) denotes point evaluation at \( s \in [a, b] \), \( \delta^{(k)}_s \) denotes \( k \)th derivative evaluation at \( s \),
\[
m_k := \int_t^b \frac{(x-t)^k}{k!} \, d\mu
\]
for $k = 1, \ldots, n - 2$ and

$$m_{n-1} := \int_t^b \frac{(x - t)^{n-1}}{(b - t)(n - 1)!} \, d\mu.$$

**NOTE 3** The main result of the next section concerns monotone functions which are not strictly monotone; obviously such functions must be constant on a subinterval of $[a, b]$. More generally (see Corollary 3) we will consider $n$-convex functions which fail to be strictly $n$-convex; that is, $n$-convex functions which are polynomial of degree $(n - 1)$ on some subinterval of $[a, b]$.

## 2 Main Results

The primary goal of this paper is to demonstrate that monotonically complemented subspaces of $C[a, b]$ are (relatively) rare. We include several corollaries of this result, including a necessary condition associated with the preservation of $n$-convexity using projections.

**THEOREM 1** For integer $d \geq 2$, let $V \subset X$ be $d$-dimensional. If $V$ contains a 2-dimensional subspace $V_2$ such that every monotone $v \in V_2$ is strictly monotone then $V$ is not monotonically complemented.

**NOTE 4** Taken in conjunction with Notes 1 and 3, Theorem 1 indicates that monotonically complemented $V$ must possess $d - 1$ linearly independent elements from $S_1$, all of which are constant on some interval.

**COROLLARY 1** For $d \geq 2$, let $V \subset X$ be an $d + 1$-dimensional Haar subspace that contains the constant functions. Then $V$ is not monotonically complemented.

**Proof.** If $V$ were monotonically complemented then there would exist a non-constant $f \in S \cap V$ that was constant on some sub-interval of $[a, b]$; this implies the existence of an element of $V$ with infinitely many zeros - a contradiction to the Haar condition. 

**COROLLARY 2** Every polynomial subspace (algebraic or trigonometric) of dimension three (or greater) fails to be monotonically complemented.
As a corollary of Theorem 1 we can easily obtain a necessary condition for the existence of an \( n \)-convex preserving projection \((n \geq 2)\) defined on \( X \). The result assumes the range of such a projection is contained in \( C^{n-1}[a,b] \). This assumption is actually more 'smoothness' than required by \( n \)-convexity, as an \( n \)-convex preserving projection need only have range in \( C^{n-2}[a,b] \). However, it appears that this stronger assumption is needed in order to obtain the corollary as an immediate consequence Theorem 1. In the following we use the fact that \( f \in C^{n-1} \) is \( n \)-convex iff \( f^{(n-1)} \) is monotone.

**Corollary 3** Let \( n \) and \( d \) be positive integers greater than or equal to 2. Let \( V \subset X \) be a \( d \)-dimensional subspace such that \( V \subset C^{n-1}[a,b] \). If \( V \) contains a 2-dimensional subspace \( V_2 \) such that every element of \( V_2 \cap S_n \) is strictly \( n \)-convex then \( n \)-convexity cannot be preserved via a projection from \( X \) onto \( V \).

**Proof.** Let \( P : X \to V \) be a projection such that \( PS_n \subset S_n \). By way of contradiction, suppose there exists 2-dimensional subspace \( V_2 \subset V \) such that every element of \( V_2 \cap S_n \) is strictly \( n \)-convex. Now note that \( P \) also defines a projection from \( X_{n-2} \) (see 4) onto \( V \) leaving invariant the cone \( S_n \cap C^{n-1} \). Using a well-known representation of elements from \( X_{n-1}^* \) it follows that there exists, for \( i = 1, \ldots, d \), \( v_i \in V \) and signed measures \( \mu_i \) on \( [a,b] \) such that

\[
P f = \sum_{i=1}^{d} \left( \int_{[a,b]} f^{(n-1)} d\mu_i \right) v_i
\]

for all \( f \in X_{n-1} \). From here, we define \( Q : X \to V^{n-1} \) by

\[
Q g = \sum_{i=1}^{d} \left( \int_{[a,b]} g d\mu_i \right) v_i^{(n-1)}
\]

where \( V^{n-1} := \{ v^{(n-1)} | v \in V \} \). Note that \( Q \) is a projection onto \( V^{n-1} \) since \( P \) is a projection onto \( V \). Moreover we claim \( Q \) preserves monotonicity; if \( g(t) \) is monotone then

\[
G(t) := \int_{a}^{t} \int_{a}^{s_1} \cdots \int_{a}^{s_{n-2}} g(s_{n-1}) ds_{n-1} \cdots ds_1
\]

is \( n \)-convex and therefore

\[
\frac{d^{n-1} PG}{dp^{n-1}}(t) = Qg \quad \text{is monotone.}
\]
But this is in contradiction to Theorem 1 since the monotone elements of $V_{2^n-1}$ are strictly monotone. Thus if there exists $V_2 \subset V$ with every element of $V_2 \cap S_n$ strictly $n$-convex then no projection onto $V$ from $X$ can preserve $n$-convexity. ■

2.1 Proof of Theorem 1

The proof of Theorem 1 relies on the following three results. Note, in particular, the first two of these are valid in a general Banach space setting.

PROPOSITION 1 Let $E$ denote a Banach space and $V \subset a$ $d$-dimensional subspace. Let $\{v_1, \ldots, v_d\}$ be a basis for $V$ and define $\mathbf{v} := (v_1, v_2, \ldots, v_d)$. For $\phi \in E^*$ let $\phi(\mathbf{v})$ denote the $d$-tuple $(\phi(v_1), \ldots, \phi(v_d))$. Then $S^*_V \subset E^*_V$ is closed if and only if $S^*_V \subset \mathbb{R}^d$ is closed (using the standard topology of $\mathbb{R}^d$), where $S^*_V := \{\phi(\mathbf{v}) \mid \phi \in S^*\}$.

**Proof.** Equip $E^*$ with its weak* topology; thus a sequence $\{(\phi_k)_{|V}\}$ in $S^*_V \subset E^*_V$ converges iff the sequence $\{\phi_k(v_i)\}$ converges for each $i = 1, \ldots, d$. From here the result follows easily. ■

NOTE 5 For reasons nearly identical to those in Note 1, we may always assume $S^*_V$ is $d$-dimensional. If $S^*_V$ has dimension $k$, $1 \leq k \leq d$, for the particular basis given by $\mathbf{v} = (v_1, \ldots, v_d)$ then there exists another basis for $V$ given by $\mathbf{w} = (w_1, \ldots, w_d)$ such that $\phi(w_j) = 0$ for $\phi \in S^*$ and $j = k+1, \ldots, d$. Thus, using the notation of Note 1 and the characterization of $S$ given in Lemma 1, we find $PS \subset S$ if and only if $P_1S \subset S$, where $P : E \to V$ has representation $P = u_1 \otimes w_1 + \cdots + u_k \otimes w_d$ and $P_1 = u_1 \otimes v_1 + \cdots + u_k \otimes v_k$. Thus we may always assume that $S^*_V$ is $d$-dimensional.

LEMMA 5 Let $E$ denote a Banach space and $S \subset E$ a pointed cone. Let $V \subset X$ be a $d$-dimensional subspace, where $d \geq 2$. Suppose $V$ contains a subspace $V_2$ of dimension 2 such that $\phi(f) > 0$ for every (nonzero) $\phi \in S^*$ and every (nonzero) $f \in V_2 \cap S$. Then $P_S(E, V) = \emptyset$.

**Proof.** Assume $P_S \neq \emptyset$ and let $P : E \to V$ such that $PS \subset S$. From Note 5 we may assume $S^*_V$ is $d$-dimensional. From Note 1, there exists basis for
$V$ which belongs to $S$; let $\{f, g, v_3, \ldots, v_d\}$ be such basis where $\{f, g\}$ form a basis for $V_2$. Let

$$
v := \begin{bmatrix}
f \\
g \\
v_3 \\
\vdots \\
v_d
\end{bmatrix}.
$$

The combination of Lemma 3 and Proposition 1 implies that

$$S^*_\v := \{\phi_\v \mid \phi \in S^*\}$$

forms a closed pointed cone in the nonnegative orthant of $\mathbb{R}^d$, where

$$\phi_\v := \begin{bmatrix}
\phi(f) \\
\phi(g) \\
\phi(v_3) \\
\vdots \\
\phi(v_d)
\end{bmatrix}.$$

Note from our hypothesis that the first two coordinates of every (nonzero) point of $S^*_\v$ are strictly positive. This, together with the fact that $S^*_\v$ is closed, implies that there exists $\theta \in \mathbb{R}$ such that for matrix

$$A := \begin{bmatrix}
\cos \theta & -\sin \theta & 0 & \cdots & \cdots & 0 \\
\sin \theta & \cos \theta & 0 & \cdots & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
& & & \ddots & & \\
0 & \cdots & \cdots & \cdots & 0 & 1
\end{bmatrix}$$

we have the cone

$$AS^*_\v := \left\{A\phi_\v \mid \phi_\v \in S^*_\v\right\}$$

in the nonnegative orthant $\mathbb{R}^n$ and tangent to the hyperplane orthogonal to $[1, 0, \cdots, 0]^T$ i.e., there exists $\psi_\v \in S^*_\v$ such that $A\psi_\v$ is nonzero but has a first coordinate of 0. But $A\psi_\v = \psi_{\v_A}$ which implies the existence of an element in $V_2 \cap S$ on which $\psi \in S^*$ vanishes. This in in contradiction to our hypothesis and thus we must have $\mathcal{P}_S = \emptyset$. \qed
LEMMA 6 Let $X = (C[a,b], \| \cdot \|_\infty)$. Let $f \in S_1 \subset X$ be strictly monotone. Then there does not exist a (non-zero) element of $S^*$ which vanishes against $f$.

**Proof.** Our approach is similar to that of Lemma 4. Let $f \in S_1$ and $\mu \in S_1^*$, $(\mu \neq 0)$, such that

$$\int_{[a,b]} f \ d\mu = 0.$$ 

We show that $f$ cannot be strictly monotone. Let $E := \text{supp}(\mu)$ and suppose $E$ is split by an interval; that is, assume there exists $\alpha < \beta$ belonging to $(a,b)$ such that $E \subset [a,\alpha) \cup (\beta,b)$. Choose $t_0, t_1 \in (\alpha,\beta)$ with $t_0 < t_1$. Using $t_0$, proceed as in (3) and define

$$\mu_i := \mu|_{E_i} + (-1)^i m \delta_{t_0} \quad (5)$$

for $i = 0, 1$ where $E_0 := E \cap [a,t_0)$, $E_1 := E \cap (t_0,b]$ and $m = \mu((t_0,b])$. From Lemma 4 we have $\mu_i \in S^*$ for each $i$ and $\mu = \mu_0 + \mu_1$. Therefore it must be the case that

$$\int_{[0,1]} f \ d\mu_0 = 0.$$ 

In an identical manner, use $t_1$ to define

$$\hat{\mu}_i := \mu|_{\hat{E}_i} + (-1)^i \hat{m} \delta_{t_1} \quad (6)$$

for $i = 0, 1$ where $\hat{E}_0 := E \cap [a,t_1)$, $\hat{E}_1 := E \cap (t_1,b]$ and $\hat{m} = \mu((t_1,b])$. As in the above, it follows that

$$\int_{[0,1]} f \ d\hat{\mu}_0 = 0.$$ 

But, from our choice of $t_0$ and $t_1$, it follows that $\hat{E}_0 = E_0$ and $\hat{m} = m$. Therefore

\[
0 = \int_{[0,1]} f \ d\mu_0 - \int_{[0,1]} f \ d\hat{\mu}_0 \\
= \int_{E_0} f \ d\mu + mf(t_0) - \left( \int_{E_0} f \ d\mu + mf(t_1) \right) \\
= m(f(t_0) - f(t_1))
\]
which is possible if and only if \( f(t_0) = f(t_1) \). Thus \( f \) cannot be strictly increasing in this case.

Suppose the support of \( \mu \) fails to be split. We now construct a measure \( \nu \), with properties similar to \( \mu \), but with split support. Choose \( t_0 < t_1 \) belonging \( E \cap (a, b) \) such that \( \mu(t_0) = \mu(t_1) = 0 \). Define \( \mu_0 \) as in (5) and \( \hat{\mu}_1 \) as in (6). Now let \( \nu := \mu_0 + \hat{\mu}_1 \); from the above we have \( \nu \in S^\ast \) and

\[
\int_{[a,b]} f \, d\nu = 0.
\]

Since the support of \( \nu \) is split by \((t_0, t_1)\) see that \( f \) cannot be strictly increasing. □

**Proof of Theorem 1.** By Lemma 6, we have \( \phi(f) > 0 \) for every (nonzero) \( \phi \in S^\ast \) and \( f \in V_2 \cap S \). Therefore \( V \) is not monotonically complemented by Lemma 5. □

**References**


