A characterization and equations for minimal shape-preserving projections

B. L. Chalmers
University of California, Riverside
D. Mupasiri
University of Northern Iowa
M. P. Prophet
University of Northern Iowa

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Abstract

Let $X$ denote a (real) Banach space and $V$ an $n$-dimensional subspace. We denote by $\mathcal{B} = \mathcal{B}(X,V)$ the space of all bounded linear operators from $X$ into $V$; let $\mathcal{P}$ be the set of all projections in $\mathcal{B}$. For a given cone $S \subset X$, we denote by $\mathcal{P}_S$ the set operators $P \in \mathcal{P}$ such that $PS \subset S$. When $\mathcal{P}_S \neq \emptyset$, we characterize those $P \in \mathcal{P}_S$ for which $\|P\|$ is minimal. This characterization is then utilized in several applications and examples.

1 Introduction

Over the last 30 years much has been written on the subject of minimal projections. Much of this work involves, in one way or another, the determination of a projection $P_{\text{min}}$ from a Banach space $X$ onto (finite-dimensional) subspace $V$ such that $\|P_{\text{min}}\|$ is minimized over all projections from $X$ onto $V$. The significance of this problem is well illustrated in [1], [3], [7], [8] and [10].

In the (frequent) setting in which subspace $V$ is finite-dimensional there is never a question of existence - such spaces are always complimented and,
moreover, a projection of minimal norm (from $X$ onto $V$) always exists. Indeed, as described in a characterization of minimality from [2], existence follows from the fact that the minimal projection problem is equivalent to a best-approximation problem in a $C(K)$ space (specifically the best-approximation of a fixed function from a linear (approximating) function-space defined on the compact set $K$). We note that this characterization has been successfully employed in a variety of settings (see e.g. [2], [5]). All of this changes, however, once we place a constraint on the projections from $X$ onto $V$; in particular if require such projections to leave invariant a specified cone, existence of such operators is immediately called into question. Furthermore, such a constraint eliminates the use of the characterization in [2] - in the language of this characterization, there is no longer an obvious linear space from which to best-approximate.

If $S \subset X$ is a cone (a convex set, closed under nonnegative scalar multiplication) and $P : X \to V$ a projection, we say $P$ is shape-preserving projection if $PS \subset S$. In this paper we use techniques from [9] to develop a characterization for minimal-norm shape-preserving projections. Not surprisingly, the theory of existence of shape-preserving projections plays an important part in this characterization. Indeed, existence and the minimal-norm characterization are so closely connected we include as part of this paper a portion of existence theory.

Specifics regarding the cones under consideration are contained in Section 2. We also include in this section properties of such cones; since they are infinite-dimensional it is necessary to verify certain basic properties. Section 3 describes a geometric equivalence to the existence of shape-preserving projections and Section 4 utilizes this condition to characterize minimal-norm shape-preserving projections. The final section demonstrates how the theory of the previous sections comes together to solve (non-trivial) problems in various classical settings. We note here that each application in Section 5 examines particular aspects of open questions in projection theory; as such, in addition to illustration, these examples are meant to serve as starting points for further investigations.

## 2 General preliminaries

Throughout this paper $X$ will denote a real Banach space with unit ball and sphere denoted by $B(X)$ and $S(X)$, respectively. For fixed positive integer
$n$, $V \subset X$ will always denote an $n$-dimensional subspace of $X$. For a given $V$ and $X$, $\mathcal{B} = \mathcal{B}(X, V)$ will denote the set of linear operators from $X$ into $V$, while $\mathcal{P} \subset \mathcal{B}$ will denote the set of all projections.

We now review some basic terminology from convex analysis. In a (real) topological vector space, a cone $K$ is a convex set, closed under nonnegative scalar multiplication. $K$ is pointed if it contains no lines. For $\phi \in K$, let $[\phi]^+ := \{\alpha \phi \mid \alpha \geq 0\}$. We say $[\phi]^+$ is an extreme ray of $K$ if $\phi = \phi_1 + \phi_2$ implies $\phi_1, \phi_2 \in [\phi]^+$ whenever $\phi_1, \phi_2 \in K$. We let $E(K)$ denote the union of all extreme rays of $K$. When $K$ is a closed, pointed cone of finite dimension we always have $K = \text{co}(E(K))$ (this need not be the case when $K$ is infinite dimensional; indeed, we note in [11] that it is possible that $E(K) = \emptyset$ despite $K$ being closed and pointed). We say that a closed, pointed cone $K$ of finite dimension is simplicial whenever the number of extreme rays of $K$ is exactly $\dim(K)$.

**DEFINITION 2.1** Let $X$ be a (fixed) Banach space and $V \subset X$ a (fixed) $n$-dimensional subspace. Let $S \subset X$ denote a closed cone. We say that $x \in X$ has shape (in the sense of $S$) whenever $x \in S$. If $P \in \mathcal{P}$ and $PS \subset S$ then we say $P$ is a shape-preserving projection; we denote the set of all such projections by $\mathcal{P}_S$. For a given cone $S$, define $S^* = \{\phi \in X^* \mid \langle x, \phi \rangle \geq 0 \ \forall x \in S\}$. We will refer to $S^*$ as the dual cone of $S$.

The cone $S^*$ will play an important role in our characterization of the minimal norm element from $\mathcal{P}_S$. We will assume throughout this paper that $S^*$ is pointed and contains at least $n + 1$ linearly independent elements. Note that $S^*$ is a weak*-closed cone which is “dual” to $S$ in the following sense.

**LEMMA 2.1** Let $x \in X$, If $\langle x, \phi \rangle \geq 0$ for all $\phi \in S^*$ then $x \in S$.

**Proof.** We prove the contrapositive; suppose $x \in X$ such that $x \notin S$. Then, since $S$ is closed and convex, there exists a separating functional $\phi \in X^*$ and $\alpha \in \mathbb{R}$ such that $\langle x, \phi \rangle < \alpha$ and

$$\langle s, \phi \rangle > \alpha \ \forall s \in S. \quad (1)$$

Note that we must have $\alpha < 0$ because $0 \in S$. In fact, for every $s \in S$ we claim

$$\langle s, \phi \rangle \geq 0 > \alpha. \quad (2)$$
To check this, suppose there exists $s_0 \in S$ such that $\langle s_0, \phi \rangle = \beta < 0$; this would imply

$$\langle \frac{\alpha}{\beta} s_0, \phi \rangle = \alpha$$

while $\frac{\alpha}{\beta} s_0 \in S$. And this is in contradiction to (1). The validity of (2) implies that $\phi \in S^*$ and this completes the proof. ■

**Lemma 2.2** Let $P \in \mathcal{B}$. Then $PS \subset S \iff P^* S^* \subset S^*$.

**Proof.** The proof is an immediate consequence of the duality equation $\langle Pf, u \rangle = \langle f, P^* u \rangle$ and Lemma 2.1. ■

Before characterizing minimal-norm elements of $\mathcal{P}_S$, we must first ensure that $\mathcal{P}_S \neq \emptyset$; we employ $S^*$ for this task. We begin by looking for (convenient) subsets of $S^*$ which can be used to 'recover' the entire cone. The following proposition describes one possible subset.

**Proposition 2.1** Every nonzero element of $S^*$ is contained in the convex hull of $\partial S^*$.

**Proof.** Let $x \in S^* \setminus \{0\}$. If $x \in \partial S^*$ then $x$ is interior to a line segment joining 0 and a positive scalar multiple of $x$ and thus we are finished. Suppose $x$ belongs to the interior of $S^*$ and let $e \in \partial S^*$. For $\alpha \in \mathbb{R}^+$, let $L_\alpha(e) \subset X^*$ denote the half-line beginning at $e$ and passing through $\alpha x$. We claim that for some $\alpha > 0$ we have $(L_\alpha(e) \cap \partial S^*) \setminus \{e\} \neq \emptyset$. Suppose this is not true; then for every $\alpha > 0$ the half-line $L_\alpha(e)$ is entirely contained in $S^*$. And since $S^*$ is closed, this implies that $L_\alpha(0)$ is a half-line entirely contained in $S^*$. But this contradicts that fact that $S^*$ is pointed and our claim is established. Let $\alpha_x > 0$ be as in the above claim and let $y_x := (L_\alpha(e) \cap \partial S^*) \setminus \{e\}$. Thus

$$\alpha_x x = \lambda e + (1 - \lambda)y_x$$

for some $0 < \lambda < 1$. Dividing both sides of 3 by $\alpha_x$ we that $x$ is on the line segment joining boundary elements $\frac{1}{\alpha_x} e$ and $\frac{1}{\alpha_x} y_x$. ■

In general, however, $\partial S^*$ is difficult to describe and thus is of limited utility in representing elements of $S^*$. Another natural subset to consider is $E(S^*)$, the set of all extreme rays of $S^*$. The following definition provides a
description of those $S^*$ for which $E(S^*)$ provides the right recovery set. In the context of our current considerations, we say a finite (possibly) signed measure $\mu$ with support $E \subset X^*$ is a generalized representing measure for $\phi \in X^*$ if $\langle x, \phi \rangle = \int_E \langle s, x \rangle \, du(s)$ for all $x \in X$. A nonnegative measure $\mu$ satisfying this equality is simply a representing measure.

**DEFINITION 2.2** We say that $S^*$ (the pointed dual cone of $S$) is simplicial if $S^*$ can be recovered from its extreme rays (i.e., $S^* = \overline{\omega(E(S^*)))}$) and the set of extreme rays form an independent set (independent in the sense that, any generalized representing measure supported on $E(S^*)$ for $\phi \in S^*$ must be a representing measure). A pointed, closed cone of finite dimension $k$ is simplicial if there exist exactly $k$ (extreme) rays of the cone whose convex hull is the entire cone.

Unless otherwise noted, $S^*$ is assumed to be simplicial. Equipped with this assumption, the following theorem provides us with an easily applied test to determine if $\mathcal{P}_S \neq \emptyset$.

**THEOREM 2.1 (see [9])** $\mathcal{P}_S \neq \emptyset$ if and only if the cone $S^*_{|V}$ is simplicial.

**NOTE 2.1** Suppose $S^*_{|V}$ is $k$-dimensional where $1 \leq k \leq n$. Choose a basis for $V$, $v_1, \ldots, v_n$ such that, for $i = 1, \ldots, n - k$, $\langle v_i, u \rangle = 0$ for all $u \in S^*$ and, for $i = n - k + 1, \ldots, n$, $v_i \in S$. With this basis, any operator $P : X \rightarrow V$ can be written $P = u_1 \otimes v_1 + \cdots + u_n \otimes v_n$ for some choice of $u_i$’s $\in X^*$, where $Pf = \langle f, u_1 \rangle v_1 + \cdots + \langle f, u_n \rangle v_n$ (for convenience, we often write $u := (u_1, \ldots, u_n) \in (X^*)^n$, $v := (v_1, \ldots, v_n)^T \in V^n$ and $Pf = \langle f, u \rangle v$). Thus we note that $P : X \rightarrow V$ is shape-preserving if and only if $P_1 : X \rightarrow V_1$ is shape-preserving where $V_1 := [v_{n-k+1}, \ldots, v_n]$ and $P_1 = u_{n-k+1} \otimes v_{n-k+1} + \cdots + u_n \otimes v_n$.

**COROLLARY 2.1** Suppose $\mathcal{P}_S \neq \emptyset$ and $S^*_{|V}$ is $k$-dimensional. Then there exists a basis $v = (v_1, \ldots, v_n)^T$ for $V$ and $u = (u_1, \ldots, u_n) \in (X^*)^n$ such that whenever $P = u \otimes v \in \mathcal{P}_S$, we have, for $i = n - k - 1, \ldots, n$, $u_i \in S^*$. Moreover, each such $u_i$ restricts to a distinct extreme ray of $S^*_{|V}$.

**Proof.** $\mathcal{P}_S \neq \emptyset$ implies that $S^*_{|V}$ has exactly $k$ edges and is expressible as

$$S^*_{|V} = \text{cone}(u_{n-k-1|V}, \ldots, u_n|V)$$
for some \((u_{n-k-1}, \ldots, u_n) =: u \in (S^*)^n\). Choose a basis \(\{v_1, \ldots, v_n\}\) for \(V\) as in Note 2.1 and define \(P := u \otimes M^{-1}v : X \rightarrow V_1\) where \(V_1 := [v_{n-k-1}, \ldots, v_n]\), \(v = (v_{n-k-1}, \ldots, v_n)^T\) and \(M = \langle v, u \rangle = \langle (v, u_j) \rangle\). Obviously \(P\) is a projection onto \(V_1\); the fact that \(P\) is a shape-preserving follows from Lemma 3 in [9]. Let \(Q := \phi \otimes M^{-1}v\) be an arbitrary projection onto \(V_1\) preserving \(S\). This implies that \(\phi \in (S^*)^n\), since \(\phi = Q^*u\) and \(Q^*u \in (S^*)^n\) by Lemma 2.2. Furthermore, \(\phi_{1v} = u_{1v}\) since \(I_n = \langle M^{-1}v, \phi \rangle = \langle M^{-1}v, u \rangle\). Therefore, the only way to construct a projection onto \(V_1\) preserving \(S\) is to “extend” the extreme rays of \(S_{1v}^*\) from \(V\) to \(X\). This observation, taken together with the concluding remark of Note 2.1, completes the proof. ■

**COROLLARY 2.2** Suppose \(\mathcal{P}_S \neq \emptyset\) and \(S_{1v}^*\) is \(n\)-dimensional. If elements of \(S^*\) have unique restrictions to \(E(S_{1v}^*)\) then \(\mathcal{P}_S\) contains a unique element.

**Proof.** This is an immediate consequence of Corollary 2.1. ■

## 3 Characterization

In order to determine minimal norm elements of \(\mathcal{P}_S\), we will cast the problem in a ‘continuous function on a compact set’ setting.

**DEFINITION 3.1** \((x, y) \in S(X^{**}) \times S(X^*)\) will be called an extremal pair for \(P \in \mathcal{P}_S\) if \(\langle P^{**}x, y \rangle = \|P\|\), where \(P^{**} : X^{**} \rightarrow V\) is the second adjoint extension of \(P\) to \(X^{**}\).

**NOTATION** Let \(\mathcal{E}(P)\) be the set of all extremal pairs for \(P\). To each \((x, y) \in \mathcal{E}(P)\) associate the rank one operator \(y \otimes x\) from \(X\) to \(X^{**}\) given by \((y \otimes x)(z) = \langle z, y \rangle x\) for \(z \in X\).

**THEOREM 3.1** (Characterization of minimal \(P\) in \(\mathcal{P}_S\)). Let \(\mathcal{P}_S\) be non-empty. Suppose that \(S_{1v}^*\) is \(k\)-dimensional, \(1 \leq k < n\) and \(S^*\) restricts uniquely to \(E(S_{1v}^*)\). Then \(P \in \mathcal{P}_S\) has minimal norm in \(\mathcal{P}_S\) if and only if the closed convex hull of \(\{y \otimes x \mid (x, y) \in \mathcal{E}(P)\}\) contains an operator carrying \(V_0\) into \(V\), where \(V_0 := V \cap ((S^*)^\perp)\). If \(k = n\) then either \(\mathcal{P}_S = \emptyset\) or \(|\mathcal{P}_S| = 1\).
Proof. In the case $k = n$, if $\mathcal{P}_S \neq \emptyset$ then $|\mathcal{P}_S| = 1$ by Corollary 2.2; thus we now assume $k < n$. Fix basis $v_1, v_2, \ldots, v_n$ for $V$ such that $V_0 := [v_1, \ldots, v_{n-k}] \in (S^*)^\perp$ and $V_1 := [v_{n-k+1}, \ldots, v_n]$. Let $P_0 = \sum_{i=1}^n u_i \otimes v_i \in \mathcal{P}_S$ where $u_i \in X^*$ for $i = 1, \ldots, n$. By Note 2.1 we have $P_1 := \sum_{i=n-k+1}^n u_i \otimes v_i \in \mathcal{P}_S$ and thus by Corollary 2.2 (replacing $n$ with $k$) we have that $P_1$ is the unique projection preserving $S$ onto $V_1$. Consequently, the functionals $u_{n-k+1}, \ldots, u_n$ appearing in the definition of $P_0$ are unique (among all possible choices of functionals defining a shape-preserving projection onto $V$ with respect to basis $v_1, \ldots, v_n$). Therefore, the problem of finding a minimal-norm element from $\mathcal{P}_S$ is equivalent to best approximating, in the operator norm, the fixed operator $P_0 \in \mathcal{P}_S$ from the space of operators $\mathcal{D} = \left\{ \sum_{i=1}^{n-k} \epsilon_i \otimes v_i \mid \epsilon_i \in (S^*)^\perp \right\} = \text{sp}\{\delta \otimes v : \delta \in (S^*)^\perp, \ v \in V_0\}$. Let $K = B(X^{**}) \times B(X^*)$ endowed with the product topology, where $B(\cdot^*)$ denotes the unit ball with its weak$^*$ topology. Associate with any operator $Q \in B$ the bilinear form $\hat{Q} \in C(K)$ via $\hat{Q}(x, y) = \langle Q^{**} x, y \rangle$, and let $\hat{\mathcal{D}} = \{\hat{\Delta} : \Delta \in \mathcal{D}\}$. Then, making use of standard duality theory for $C(K)$, $K$ compact (see e.g., [12], Theorem 1.1 (p. 18) and Theorem 1.3 (p. 29)), we have that $\hat{P} = \hat{P}_0 - \hat{\Delta}_0$ is of minimal norm if and only if there exists a finite, non-zero (total mass one) signed measure $\hat{\mu}$ supported on the critical set

$$\mathcal{C}(\hat{P}) = \{(x, y) \in S(X^{**}) \times S(X^*) : |\hat{P}(x, y)| = \|\hat{P}\|_\infty\}$$

such that $\text{sgn} \hat{\mu}\{(x, y)\} = \text{sgn} \hat{P}(x, y)$ and $\hat{\mu} \in \hat{\mathcal{D}}^\perp$, i.e.,

$$0 = \int_{\mathcal{C}(\hat{P})} \hat{\Delta} \, d\hat{\mu} \quad \text{for all } \hat{\Delta} \in \hat{\mathcal{D}}.$$

But now, since any $\hat{Q} \in \{\hat{P}\} \cup \hat{\mathcal{D}}$ is a bilinear function, we can replace the signed measure $\hat{\mu}$, supported in $\mathcal{C}(\hat{P})$, by a positive measure $\mu$ supported on $\mathcal{E}(P) \subset \mathcal{C}(\hat{P})$ by noting that

$$\mathcal{C}(\hat{P}) = \{(x, \pm y) : (x, y) \in \mathcal{E}(P)\},$$

and setting

$$\mu\{(x, y)\} = |\hat{\mu}|\{(x, y), (x, -y)\}.$$

For then $\text{sgn} \mu\{(x, y)\} = \text{sgn} \hat{P}(x, y) = 1$, for $(x, y) \in \mathcal{E}(P)$ and

$$0 = \int_{\mathcal{E}(P)} \hat{\Delta} \, d\mu \quad \text{for all } \Delta \in \mathcal{D},$$

7
since
\[
\int_{C(\tilde{P})} \Delta d\mu = \int_{\{x, y \in \mathcal{E}(P) \mid k \in \{0, 1\}\}} \Delta(x, (-1)^k y) d\mu(x, (-1)^k y)
\]
\[
= \int_{\{x, y \in \mathcal{E}(P) \mid k \in \{0, 1\}\}} (-1)^k \Delta(x, y) (-1)^k d|\mu|(x, (-1)^k y)
\]
\[
= \int_{\mathcal{E}(P)} \Delta d\mu.
\]
Hence,
\[
0 = \int_{\mathcal{E}(P)} \Delta d\mu = \int_{\mathcal{E}(P)} \langle \Delta^{**}, x, y \rangle d\mu(x, y)
\]
\[
= \int_{\mathcal{E}(P)} \langle x, \delta \rangle \langle v, y \rangle d\mu(x, y)
\]
\[
= \left< \int_{\mathcal{E}(P)} \langle v, y \rangle x d\mu(x, y), \delta \right>
\]
for all $\Delta = \delta \otimes v$ ($\delta \in (S^*)^\perp$, $v \in V_0$), where, for $z \in X$, $\int_{\mathcal{E}(P)} \langle z, y \rangle x d\mu(x, y)$ is the element $w \in X^{**}$ defined by $\langle x^*, w \rangle = \int_{\mathcal{E}(P)} \langle z, y \rangle \langle x^*, x \rangle d\mu(x, y)$ for all $x^* \in X^*$. $P$ is minimal, therefore, if and only if $\int_{\mathcal{E}(P)} \langle v, y \rangle x d\mu(x, y) \in (V_n^\perp)^\perp = V_n$, i.e., if and only if there exists an operator (from $X$ into $X^{**}$)
\[
E_P = \int_{\mathcal{E}(P)} y \otimes x d\mu(x, y) : V_0 \to V_n.
\]

4 Applications

We now apply the above minimization theory in various classical settings. As is clear from the development, minimal shape-preserving projection theory is closely connected with existence theory. Existence of shape-preserving projections relies on the relationships between three ‘players’: the overspace $X$, subspace $V \subset X$ and the shape $S \subset X$ to be preserved (or equivalently $S^* \subset X^*$). As the following examples illustrate, relatively small changes in the triple $(X, V, S)$ greatly impact $\mathcal{P}_3$. 

8
EXAMPLE 4.1 Let \( X = C^2[0,1] \) and \( V = \Pi_4 \) - the space of 4th-degree algebraic polynomials considered as a subspace of \( C^2[0,1] \). Let \( S \subset X \) denote the cone of convex functions. In this case \( S^* \) is simplicial with the set of extreme rays \( E(S^*) = \{ [\delta_i^+] \}_{i \in [0,1]} \) where \( \delta_i^+ \in X^* \) denotes 2nd-derivative evaluation at \( t \). As verified in \([5]\), this combination of \( X \), \( V \) and \( S \) forces \( \mathcal{P}_S = \emptyset \). However, by changing to \( V = \Pi_3 \), we find that \( S^*_v \) is simplicial and thus \( \mathcal{P}_S \neq \emptyset \).

EXAMPLE 4.2 Let \( X = C[0,1] \) and \( V = \Pi_2 \). Let \( S \subset X \) denote the cone of nonnegative functions. In this case \( S^* \) is simplicial with the set of extreme rays \( E(S^*) = \{ [\delta_i] \}_{i \in [0,1]} \) where \( \delta_i \in X^* \) denotes point-evaluation at \( t \). It is immediately clear that \( S^*_v \) fails to be simplicial; indeed each extreme ray of \( S^* \) restricts to a unique extreme ray of \( S^*_v \). Thus \( \mathcal{P}_S = \emptyset \). Now consider the following small variation: let \( \phi \in X^* \) denote any functional such that

\[
\langle 1, \phi \rangle = \alpha, \quad \langle x, \phi \rangle = \beta \quad \text{and} \quad \langle x^2, \phi \rangle = 0
\]

where \( \beta / \alpha \geq 1/2 \). Let \( S_1^* = \overline{\sigma(S^* \cup [\phi]_+)} \) and define \( S_1 := \{ x \in X \mid \langle x, u \rangle \geq 0 \ \forall u \in S_1^* \} \). It is easy to verify that \( (S_1)_v \) is simplicial and therefore the \( S_1 \) shape can be preserved by a projection onto \( V \); i.e., \( \mathcal{P}_{S_1} \neq \emptyset \).

EXAMPLE 4.3 Let \( X = C[0,1] \) and \( V = \Pi_2 \). Let \( S \subset X \) denote the cone of non-decreasing functions. In Lemma 4 of \([9]\), it is demonstrated that, regardless of whether \( S^* \) is simplicial, the cone \( S^*_v \) must be closed in order for \( \mathcal{P}_S \neq \emptyset \). We now show that this cone fails to be closed. Consider \( S^*_v \): since every element of this cone vanishes on the identically 1 function, we can regard \( S^*_v \) as a subset of \( \mathbb{R}^2 \) by associating each \( \phi_v \in S^*_v \) with the 2-tuple \( (\langle x, \phi \rangle, \langle x^2, \phi \rangle) \). We claim that the ray determined by \( e_1 := (1,0) \) does not belong to the cone. Suppose, to the contrary, that there exists \( \phi \in S^* \) such that \( \phi_v = (1,0)^T \). Let \( m \) be an arbitrary positive integer and consider the function \( F(t) := mt^2 - G(t) \) where \( G(t) \) is any \( C^1 \) function such that \( 0 \leq G'(t) \leq 2mt \) for all \( t \in [0,1] \). \( F \) is monotone so \( \langle F, \phi \rangle \geq 0 \); but \( G \) is also monotone and \( \phi \) vanishes on \( t^2 \). The only possibility then is that \( \phi \) vanishes on \( G \). However, vanishing on all such \( G \) leads quickly to the conclusion that \( \phi \) is unbounded. Therefore the ray determined by \( e_1 \) does not belong to the cone and, moreover, the cone is not closed. Therefore \( \mathcal{P}_S = \emptyset \). However, if we change to \( X = C^1[0,1] \) (but keep \( V = \Pi_2 \) and \( S \subset C^1[0,1] \) as the cone of non-decreasing functions) we find \( S^*_v \) to be simplicial and thus \( \mathcal{P}_S \neq \emptyset \).
4.1 $C^m[a,b]$

For fixed positive integer $m$ let $X$ denote the $m$-th continuously differentiable functions, $C^m[a,b]$, normed by $\|f\| := \max_{i=0\ldots m}\{\|f^{(i)}\|_\infty\}$. In this setting, note that $\delta^{(k)}_t$, $k$-th derivative evaluation at $t$ belongs to sphere of $X^*$ whenever $0 \leq k \leq m$ and $t \in [a,b]$. Moreover, for fixed $k$, the cone $S^* := \overline{\text{cone}}(\{\delta^{(k)}_t\}_{t \in [a,b]})$ is simplicial, as per Definition 2.2, since $E_0 = \{\delta^{(k)}_t\}_{t \in [a,b]}$. We refer to the corresponding cone $S \subset X$ as the set (or cone) of $k$-convex functions. In [5], with $V := \Pi_m$ (the $m$-th degree algebraic polynomials) it is shown that $\mathcal{P}_S = \emptyset$ if and only if $k < m - 1$. For example, there is no monotonicity-preserving (1-convex preserving) projection from $X$ onto the $\Pi_3$. In the $k = m - 1$ case, the cone $S^*_{|V}$ is simplicial - 2-dimensional and closed. This implies that, of the $m+1$ functionals needed to define a minimal element from $\mathcal{P}_S$, 2 of these functionals will be given by the extreme rays of $S^*_{|V}$ and the remaining $m - 1$ must be determined. That is, the dimension of $V_0$ (as in Theorem 3.1) is $m - 1$. Theorem 4.2 in [5] identifies a minimal element of $\mathcal{P}_S$ (with norm 3/2 for every $k$) by constructing an $E_P$ operator mapping $V_0$ into $V$.

The results in [5] are exclusively related to the case $k = m - 1$, where the cone $S^*_{|V}$ is 2-dimensional (and closed) and hence automatically simplicial (the case $k = m$ is trivial in the sense that $\dim (S^*_{|V}) = 1$). One direction of generalization (that results in higher dimensional cones $S^*_{|V}$) is to seek preservation of “multi-convex” shapes, in the following sense. Using the notation of [4], let $\sigma = \{\sigma_i\}_{i=0,\ldots,m}$ be an $(m+1)$-tuple with $\sigma_i \in \{-1,0,1\}$ and define $S_{\sigma} := \{f \in X | \sigma_i f^{(i)} \geq 0, \ i = 0,\ldots , m\}$. With $V = \Pi_m$ fixed, one may look for all $\sigma$ such that $\mathcal{P}_{S_{\sigma}} \neq \emptyset$. The following example illustrates the case of $m = 3$ and $\sigma = (0,1,1,0)$, in which the associated cone $S^*_{|V}$ is 3-dimensional and simplicial.

**EXAMPLE 4.4** Let $X = C^2(0,1]$ and let $V = \Pi_3$ with fixed basis (vector) $v = [1,x,x^2,x^3]^T$. Let

$$S^* := \overline{\text{cone}}\left(\{\delta^i_0\}_{0 \leq i \leq 3} \cup \{\delta^i_1\}_{0 \leq i \leq 3}\right).$$

Thus $S \subset X$ is the cone of convex, monotone-increasing functions. Note that for $t \in [0,1]$ we have

$$(\delta^i_t)_v = (1-t)(\delta^i_0)_v + t(\delta^i_1)_v$$

Thus $S \subset X$ is the cone of convex, monotone-increasing functions. Note that for $t \in [0,1]$ we have

$$(\delta^i_t)_v = (1-t)(\delta^i_0)_v + t(\delta^i_1)_v$$

Thus $S \subset X$ is the cone of convex, monotone-increasing functions. Note that for $t \in [0,1]$ we have

$$(\delta^i_t)_v = (1-t)(\delta^i_0)_v + t(\delta^i_1)_v$$
and

$$(\delta_i')_{|V} = (\delta_0')_{|V} + (t - \frac{t^2}{2})(\delta_0'')_{|V} + \frac{t^2}{2}(\delta_i'')_{|V}.$$  

Thus $S^*_{|V}$ is a (3-dimensional) simplicial cone with extreme rays $[(\delta_0')_{|V}]^+$, $[(\delta_0'')_{|V}]^+$ and $[(\delta_i'')_{|V}]^+$. By Theorem 3.1, $\mathcal{P}_S \neq \emptyset$. Moreover, by Corollary 2.1, every $P = \sum_{i=1}^4 u_i \otimes x^{i-1} = u \otimes v \in \mathcal{P}_S$, must have $u_2 = \delta_0'$, $u_3 = \frac{1}{2}\delta_0''$ and $u_4 = \frac{1}{6}(\delta_1'' - \delta_0'')$. Thus

$$P = u_1 \otimes 1 + \delta_0' \otimes x + \frac{1}{2}\delta_0'' \otimes x^2 + \frac{1}{6}(\delta_1'' - \delta_0'') \otimes x^3 \quad (4)$$

for some $u_1 \in X^*$. We now proceed with the construction of $u_1$ so that $P$ is a minimal norm element of $\mathcal{P}_S$. In this process we will demonstrate how the theory of Theorem 3.1 guides the construction. We begin by identifying extremal pairs for $P$. Recall the $P \in \mathcal{P}_S$ is minimal if and only there is a convex combination of extremal pairs (denoted by $E_F$) mapping $V_0$ into $V$, where $V_0$ is the 1-dimensional space spanned by $v(t) = 1$. Consider possible extremal pairs of the form $(F_0, \delta_i)$, $(F_1, \delta_i')$ and $(F_2, \delta_i'')$, where $F_i \in S(X^*)$ for $i = 0, 1, 2$. From the above forms of $P$, we see for $f \in S(X)$, that

$$|(Pf)'(x)| = |(1 - x)f''(0) + f'(1)| \leq 1$$

and

$$|(Pf)''(x)| = |(-\frac{1}{2}x^2 + x)f''(0) + \frac{x^2}{2}f''(1) + f'(0)| \leq 2. \quad (5)$$

From this we conclude that, while no extremal pair of the form $(F_2, \delta_i'')$ can exist, (we assume the norm of the minimal shape-preserving projection will exceed 1) we may have an extremal of the form $(F_1, \delta_i')$. Before attempting to construct such a pair, let us briefly consider elements in $X^{**}$. Let $x_n$ be a sequence of functions in $S(X)$ such that the set

$$M = \{f \in X^* \mid \lim_{n \to \infty} \langle x_n, f \rangle \text{ exists} \}$$

is non-empty. $M$ is a subspace of $X^*$. Define on $M$ the linear functional $F : M \to \mathbf{R}$ by

$$\langle f, F \rangle = \lim_{n \to \infty} \langle x_n, f \rangle$$

Note $\|F\| = 1$. By the Hahn-Banach extension theorem, extend $F$ to all $X^*$ and thus $F \in S(X^{**})$. Of course, we don’t know the representation of $F$ off $M$.  

11
With this construction in mind, we now define a sequence \( \{x_n\} \) in \( X \) (we choose the following representation due to its utility); let

\[
    x_n(t) := \int_0^t z_n(x) \, dx
\]

where

\[
    z_n(x) := \begin{cases} 
        \frac{\frac{n-2}{2} x + \frac{n^2-3n-1}{2n^2}}{\frac{n^2}{n^2-2n+1}} x^2 + \frac{2(n^2-n-2)}{n^2-2n+1}, & 0 \leq x < 1/n \\
        \frac{2n^3-5n^2+n+4}{n^2-2n+1} x^2 - \frac{3n^3-8n^2+8+n}{n^2-2n+1} x + \frac{7n^3-18n^2-n+20}{4n(n^2-2n+1)}, & x \geq 1/2 
    \end{cases}
\]

For \( n \geq 3 \), straightforward calculations verify the following important properties of the functions \( x_n(t) \):

\[
    \|x_n\| \in S(X), \quad x'_n(0) = \frac{(n-2)(n+1)}{n(n-1)^2}, \quad x''_n(0) = x''_n(1) = 1. \tag{6}
\]

Now note that the subspace \( M = \{f \in X^* \mid \lim_{n \to \infty} \langle x_n, f \rangle \text{ exists} \} \) contains all second derivative point-evaluations since

\[
    \lim_{n \to \infty} \langle x_n, \delta'_t \rangle = \begin{cases} 1 & t = 0 \\
        -2t & 0 < t < 1/2 \\
        4t - 3 & 1/2 \leq t \leq 1.
    \end{cases}
\]

Thus we may associate \( \{x_n\} \) with a functional \( F_1 \in S(X^{**}) \) and consider the pair \((F_1, \delta'_1)\) as acting on \( P \). Using the continuity of \( P^{**} \) and equations (5) and (6) we have

\[
    \langle P^{**} F_1, \delta'_1 \rangle = \langle \lim_{n \to \infty} P x_n, \delta'_1 \rangle = \lim_{n \to \infty} \langle P x_n, \delta'_1 \rangle = \lim_{n \to \infty} \left( (-\frac{1}{2} + 1)x''_n(0) + \frac{1}{2} x''_n(1) + x'_n(0) \right) = 2
\]

Furthermore, if we were to define \( E_P := F_1 \otimes \delta'_1 \) then we would immediately obtain \( E_P : V_0 \to V \) since \( E_P(1) = 0 \). Thus, recalling the form of \( P \) given in (4), we see that if we can choose \( u_1 \in X^* \) such that, for all \( F_0 \in S(X^{**}) \) and all \( t \in [0, 1], \ (F_0, \delta_t) \) fails to be an extremal pair for \( P \) (i.e., if \( |\langle P^* F_0, \delta_t \rangle| < 2 \)
then indeed \((F_1, \delta_1)\) will be extremal for such a \(P\). Moreover we will know \(\|P\|\) minimal. To this end, consider

\[
u_1 := \frac{1}{2} \delta_0 + \delta_1 - \frac{1}{2} \delta'_0 - \frac{1}{6} \delta''_0 - \frac{1}{12} \delta''''_0.
\]

For this choice of \(u_1\) we have, for each \(f \in S(X)\),

\[
(Pf)(x) = \frac{1}{2} f(0) + \frac{1}{2} f(1) + \left(x - \frac{1}{2}\right) f'(0) + \left(\frac{1}{6} x^3 + \frac{1}{2} x^2 - \frac{1}{6}\right) f''(0) + \left(\frac{1}{6} x^3 - \frac{1}{12}\right) f''''(1).
\]

From (7) it follows that

\[
\begin{align*}
|Pf(x)| &\leq 1 + \left|x - \frac{1}{2}\right| + \left|\frac{1}{6} x^3 + \frac{1}{2} x^2 - \frac{1}{6}\right| + \left|\frac{1}{6} x^3 - \frac{1}{12}\right| \\
&\leq 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{12} \\
&< 2.
\end{align*}
\]

Thus, a minimal norm element of \(P_S\) is given by

\[
P = \left(\frac{1}{2} \delta_0 + \frac{1}{2} \delta_1 - \frac{1}{2} \delta'_0 - \frac{1}{6} \delta''_0 - \frac{1}{12} \delta''''_0\right) \otimes 1 + \delta'_0 \otimes x + \frac{1}{2} \delta''_0 \otimes x^2 + \frac{1}{6} (\delta''''_1 - \delta''''_0) \otimes x^3
\]

with \(\|P\| = 2\).

### 4.2 \(C'[a, b]\)

There are natural 'shapes' in the Banach space setting \(X = C[a, b]\) (equipped with the supremum norm) that one may look to preserve using a projection. Among these is the cone of monotone (increasing) functions, defined in the following way. Let \(S^* \subset X^*\) denote the weak-* closure of the cone generated by all (forward) differences; i.e.,

\[
S^* := \overline{\text{cone}} \left\{ \delta_{t_2} - \delta_{t_1} \mid a \leq t_1 < t_2 \leq b \right\}.
\]

Clearly the induced cone \(S \subset X\) contains exactly the monotone functions. It is somewhat surprising that this shape is particularly difficult to preserve onto finite-dimensional dimensional subspaces. A leading cause of this is the 'flatness' of the \(S^*\) cone, as now described.
**Lemma 4.1 (see [11])** The weak* closed, pointed cone $S^*$ has no extreme rays.

The unusual structure of this $S^*$ is revealed by combining Lemmas 2.1 and 4.1: on one hand, we see that every element of $S^*$ is on a line segment joining 2 boundary points; on the other hand, we see that no boundary point belongs to an extreme ray - every boundary point is on a line segment joining 2 distinct boundary points.

We say that subspace $V \subset X$ is **monotonically complemented** if there exists a monotonicity preserving projection onto $V$. The following describes a large class of subspaces that fail to be monotonically complemented.

**Corollary 4.1 (see [11])** Let $V \subset C^0[a,b]$ be a finite-dimensional subspace of $X$ such that $V' := \{v'(x) \mid v(x) \in V\} \subset X$ contains a Haar space of dimension greater than or equal to 2. Then $V$ is not monotonically complemented.

The above demonstrates, for example, that for $n \geq 2$ the subspace of $n$-degree algebraic polynomials $\Pi_n$ is not monotonically complemented. In the following example we identify a sequence of 3-dimensional monotonically complemented subspaces $V_k$ that 'converge' to $\Pi_2$. For each $k$ find, we employ the theory from Section 4 and identify a projection which is minimal among a class of monotonicity-preserving projections.

**Example 4.5** For $k \geq 2$ define $V_k := [1, x, v_k(x)]$ where

$$v_k(x) := \begin{cases} 0 & \text{if } x \leq 1/k, \\ \frac{k}{k-2}(x - 1/k)^2 & \text{if } 1/k \leq x \leq (k - 1)/k, \\ 2x - 1 & \text{if } x \geq (k - 1)/k. \end{cases}$$

For example, with $k = 5$, we plot both $x^2$ (the thicker curve) and $v_k(x)$.

Let $\mathbf{v} = (1, x, v_k(x))^T$. Notice that $V_0 := V_k \cap (S^*)^\perp = [1]$. Due to the 'flatness' of $v_k$ on the intervals $[0,1/k]$ and $[(k - 1)/k, 1]$, we find that $S^*_{\mid V_k}$ is a (2-dimensional) simplicial cone. The extreme rays of this cone are (non-uniquely) generate by $\Delta_{0\mid V_k}$ and $\Delta_{1\mid V_k}$ where

$$\Delta_0 = k(\delta_{1/k} - \delta_0) \quad \text{and} \quad \Delta_1 = k(\delta_1 - \delta_{(k-1)/k}).$$

14
Corollary 2.1 guarantees the existence of a projection $P = u \otimes v$ such that $P \in \mathcal{P}_S$. Moreover, the proof of the Corollary indicates that $u_1$ and $u_2$ must be chosen as specific linear combinations of elements from $S^*$ that agree with $\Delta_0$ and $\Delta_1$, respectively, on $V_k$. Note that we do not have unique restrictions to the extreme rays of $S^*_{V_k}$ . Nonetheless, we may still apply the characterization given in Theorem 3.1 by fixing

$$u_1 := \Delta_0 \quad \text{and} \quad u_2 := \frac{\Delta_1 - \Delta_0}{2} \quad (8)$$

and seek the minimal monotonicity-preserving projection onto $V_k$ with the $u_1$ and $u_2$ as in (8). To this end, consider

$$P := (\delta_0 + \epsilon_k) \otimes 1 + u_1 \otimes x + u_2 \otimes v_k$$

where

$$\epsilon_k = \frac{1}{4} \left( \frac{k(k-2)}{k-1} \delta_0 - \frac{k^2}{k-1} \delta_{1/k} + \frac{k^2}{k-1} \delta_{(k-1)/k} \right).$$

We claim $P$ has minimal norm (among all those projections onto $V_k$ which preserve $S$ and satisfy (8)). To see this we note that the norm of $P$ can be obtained via a Lebesgue-function approach; that is, $\|P\|$ is obtained by
maximizing the quantity \( \sup_{f \in B(X)} |Pf(x)| \) over \( x \in [0, 1] \). From the definition of \( v_k \), we see that we can consider the 3 cases \( x \in [0, 1/k], \ x \in [1/k, (k-1)/k] \), and \( x \in [(k-1)/k, 1] \). However, \( v_k = 0 \) on \([0, 1/k]\) (which causes \( \sup_{f \in B(X)} |Pf(x)| = 1 \) on this interval) and consequently we need only consider the latter two. Elementary calculus shows that, for \( x \in [(k-1)/k, 1] \), we have
\[
\sup_{f \in B(X)} |Pf(1)| \geq \sup_{f \in B(X)} |Pf(x)|
\]
and, for \( x \in [1/k, (k-1)/k] \) we have
\[
\sup_{f \in B(X)} |Pf(1/k)| \geq \sup_{f \in B(X)} |Pf(x)|.
\]
A direct calculation reveals
\[
Pf(1) = \frac{-4k + 4 + k^2}{4(k-1)}f(0) - \frac{-k^2 + 2k}{4(k-1)}f(1/k)
\]
\[
\quad - \frac{k^2 - 6k + 4}{4(k-1)}f((k-1)/k) - \frac{-k^2 + 4k - 4}{4(k-1)}f(1)
\]
and
\[
Pf(1/k) = \frac{k^2 - 2k}{4(k-1)}f(0) + \frac{-k^2 + 4k - 4}{4(k-1)}f(1/k)
\]
\[
\quad + \frac{k^2}{4(k-1)}f((k-1)/k) + \frac{-k^2 + 2k}{4(k-1)}f(1).
\]
The quantity \( Pf(1) \) is maximized when
\[
f(0) = -1, \ f(1/k) = 1, \ f((k-1)/k) = -1, \ f(1) = 1 \tag{9}
\]
and \( Pf(1/k) \) is maximized for
\[
f(0) = 1, \ f(1/k) = -1, \ f((k-1)/k) = 1, \ f(1) = -1. \tag{10}
\]
Moreover, we find that these maximized quantities are equal with common value \( k - 1 \); i.e.,
\[
\sup_{f \in B(X)} |Pf(1)| = \sup_{f \in B(X)} |Pf(1/k)| = k - 1.
\]
This allows us to define 2 extremal pairs. Let \( F, G \in B(X) \) be the piecewise linear functions such that \( F \) interpolates as in (9) and \( G \) interpolates as in (10). Then

\[
(\delta_1, F) \text{ and } (\delta_{(k-1)/k}, G)
\]

are extremal pairs for \( P \). Let \( E_P = \delta_1 \otimes F + \delta_{(k-1)/k} \otimes G \); according to Theorem 3.1 we must show that \( E_P : V_0 \to V_k \), where \( V_0 = [1] \). But

\[
E_P(1) = \langle 1, \delta_1 \rangle F + \langle 1, \delta_{(k-1)/k} \rangle G = F + G = 0 \in V_0
\]

and thus \( P \) is minimal.

References


