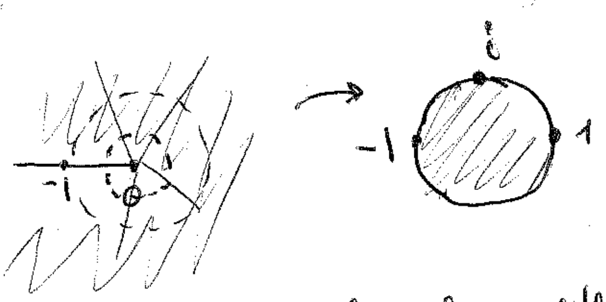
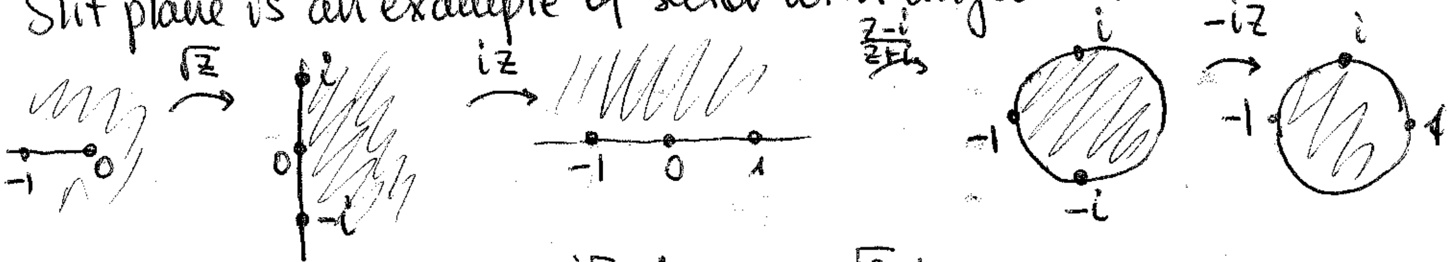
$\boxed{XI.1}$ #2

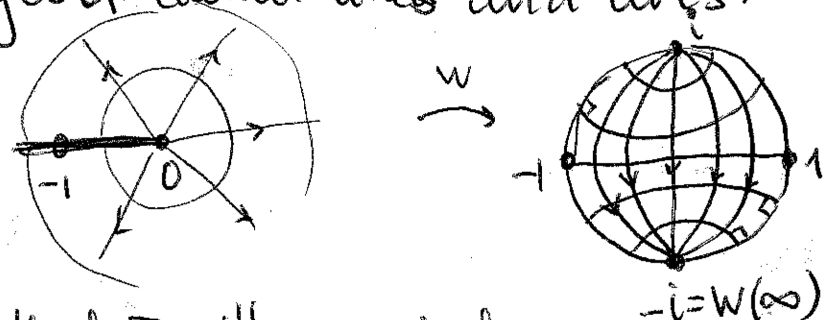


Slit plane is an example of sector with angle 2π.



Composition : $W(z) = -i \dfrac{i\sqrt{z}-i}{i\sqrt{z}+i} = -i \dfrac{\sqrt{z}-1}{\sqrt{z}+1}$

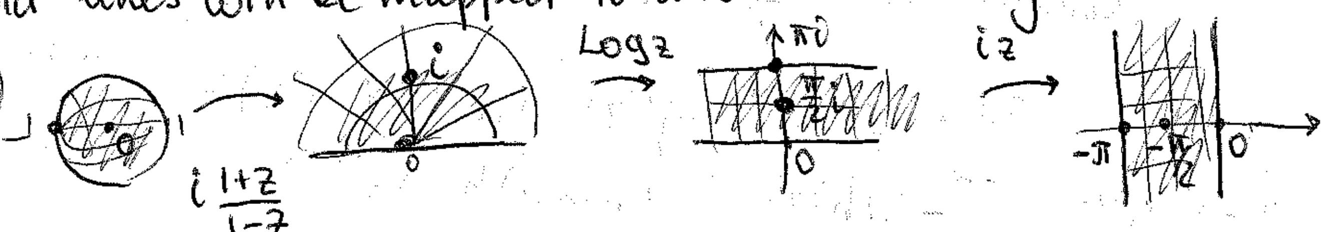
images of radial lines and arcs:

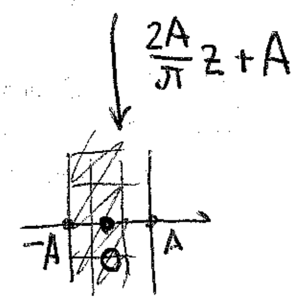


Note that $\sqrt{z}$ will map circles to semicircles, centered at 0, and radial lines to rays from 0, and all other maps are LFT, so they will map lines and circles to lines and circles

So circles, centered at 0 will be mapped to arcs of circles (and one line segment), orthogonal to the boundary of the disk, and lines will be mapped to arcs connecting i to $-i = w(\infty)$

#3





Composition $\dfrac{2Ai}{\pi} \operatorname{Log}\left(i\dfrac{1+z}{1-z}\right) + A = w(z)$

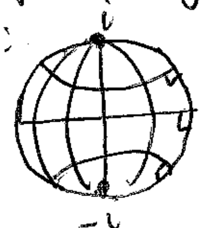
$$w'(z) = \frac{2Ai}{\pi} \cdot \frac{1-z}{1+z} \cdot \frac{2}{(1-z)^2}$$

$$w'(0) = \frac{2Ai}{\pi} \cdot 2 = \frac{4Ai}{\pi}, \quad \operatorname{Arg}(w'(0)) = \frac{\pi}{2}$$

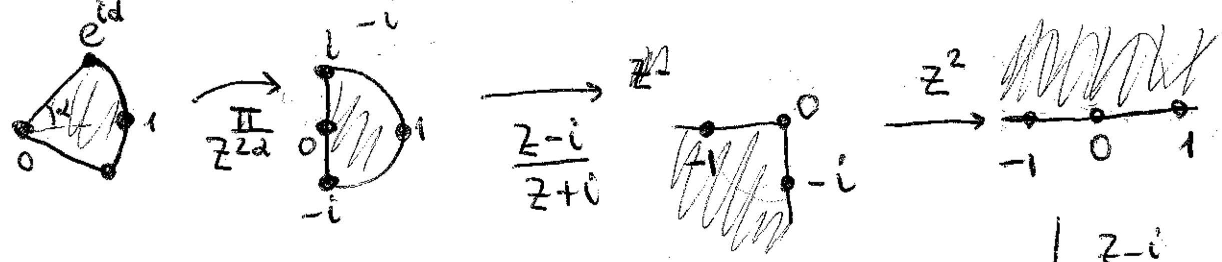
Need to adjust the argument by precomposing w with rotation by $-\frac{\pi}{2}$, i.e. by multiplication by $-i$ (and it does not change 0).

So corrected map is $\frac{2Ai}{\pi}\,\mathrm{Log}\left(i\,\frac{1-iz}{1+iz}\right)+A$.

Preimages of lines:



#7



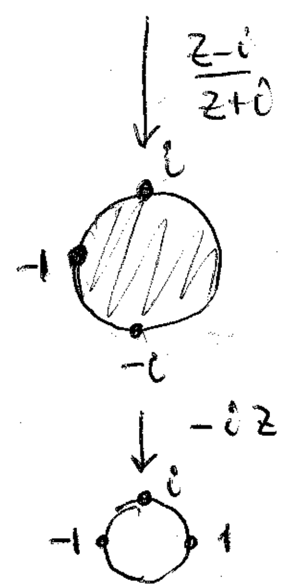
Now composition maps

$0 \mapsto -i$

$1 \mapsto i$

$e^{i\alpha} \mapsto -1$

Need to adjust by rotating by $-\frac{\pi}{2}$, i.e. multiplication by $-i$



Total map

$$W(z) = -i\left(\frac{\left(\frac{z^{\pi/2\alpha}-i}{z^{\pi/2\alpha}+i}\right)^2-i}{\left(\frac{z^{\pi/2\alpha}-i}{z^{\pi/2\alpha}+i}\right)^2+i}\right)$$ (don't simplify)

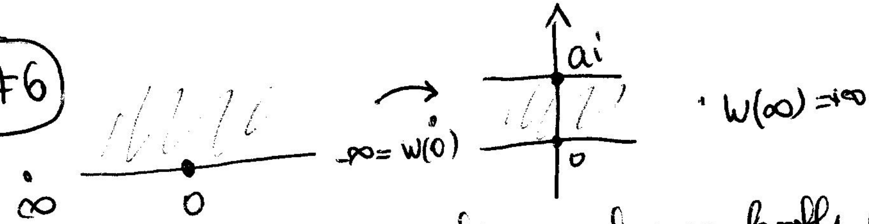
$\boxed{\text{XI. 2}}$ (#2) Consider $f(z)=\frac{\varphi(z)}{r}$ conformal, $|f(z)|=\left|\frac{\varphi(z)}{r}\right|<1$ for $z\in\mathcal{D}_r$.

$\boxed{\text{XI. 3}}$ (#1) $g(z)=w_0+h(z)\cdot z^{-\alpha}$ $\quad g'(z)=h'(z)z^{-\alpha}+h(z)(-\alpha)z^{-\alpha-1}$

$$g''(z)=h''(z)\cdot z^{-\alpha}-2\alpha h'(z)z^{-\alpha-1}+\alpha(\alpha+1)h(z)z^{-\alpha-2}$$

$$\frac{g''(z)}{g'(z)}=\frac{\alpha(\alpha+1)h(z)\frac{1}{z^2}-2\alpha h'(z)\frac{1}{z}+h''(z)}{-\alpha h(z)\frac{1}{z}+h'(z)}=-\frac{\alpha+1}{z}+\text{higher order terms in }\frac{1}{z}.$$

XI.3 #6



Polygonal domain has two vertices at $\pm\infty$, both with angle 0. ∞ is mapped to $+\infty$ does not add a term, so

$g'(z) = A(z-0)^{0-1} = \frac{A}{z}$, $g(z) = A\,\mathrm{Log}\, z + B$, $A, B \in \mathbb{C}$.

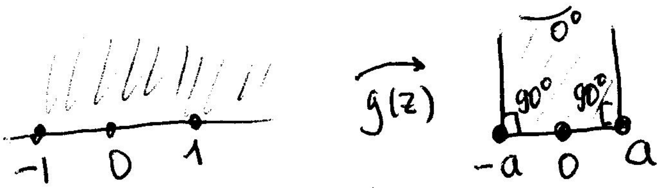
$g(1) = A \cdot 0 + B$, so $B = g(1)$.

Also, the image of $A \log z$ is horizonal strip of width $A\pi$, and $0 \mapsto -\infty$ and $\infty \to +\infty$ for $A > 0$.

Then $A\pi = a$, $A = a/\pi$, $g(z) = \frac{a}{\pi} \mathrm{Log}\, z + g(1)$.

(Note that we can select $g(1)$, this is the third real parameter).

#7



polygonal domain has 3 vertices: at $\pm a$ with angle $\frac{\pi}{2}$ and at ∞ with angle 0. By symmetry we can assume $g(\infty) = \infty$; so the vertex at ∞ will not add a term to S.C. formula.

So $a_1 = -1$, $a_2 = 1$, $\alpha_1 = \alpha_2 = \frac{1}{2}$

$g'(z) = A(z+1)^{-1/2}(z-1)^{-\frac{1}{2}} = A \frac{1}{\sqrt{z^2-1}}$

$g(z) = \int A \frac{1}{\sqrt{z^2-1}}\, dz = \frac{A}{i} \int \frac{dz}{\sqrt{1-z^2}} = c \arcsin z + b$, $c, b \in \mathbb{C}$

↑ principal branch

$g(0) = 0 + b = 0$, so $b = 0$

$g(\pm 1) = \pm c \frac{\pi}{2} = \pm a$, $c = \frac{2a}{\pi}$ $g(z) = \frac{2a}{\pi} \arcsin z$

XI.5 #1

$f_n(z)$ are uniformly bounded on D, $z_0 \in D$; $\forall m$ $f_n^{(m)}(z_0) \xrightarrow[n\to\infty]{} 0$.

By Montel's, $\exists$ subsequence $f_{n_k} \to f(z)$ normally on D.

Then $f_{n_k}^{(m)}(z_0) \to f^{(m)}(z_0) = 0$, so $f(z)$ has Taylor series expansion near z_0 identically 0, $f(z) \equiv 0$ in the neighborhood of z_0, by uniqueness, $f(z) \equiv 0$ on D.

Assume $f_n(z)$ does not converge to 0 on D; so $\exists n_j \to \infty$ s.t. $\exists A > 0$ $|f_{n_j}(z)| \geq A$. Then by Montel's again, we can extract a subsequence of f_{n_j} s.t. ft converges to 0 normally on $\mathcal{D}$, contradiction. So $f_n(z) \to 0$ normally on $\mathcal{D}$.

#3 $f(z)$ is bounded analytic on $\{-1 < \Im z < 1\}$, $f(x) \xrightarrow[x\to\infty]{} 0$

Consider $f_n(z) = f(z+n)$, analytic and uniformly bounded on the same strip.

By Montel's, $\exists$ subsequence $f_{n_k}(z) \to g(z)$ uniformly on compact subsets of strip. Fix $\varepsilon > 0$, consider compact subset $\{x=0, y\in[-1+\varepsilon, 1-\varepsilon]\}$ Then $f_{n_k}(iy) = f(iy+n_k) \to g(iy)$ uniformly in y.

$f(n_k) \to g(0) = 0$. Similarly, we can change x and have $g(x)=0$ for large set of x, by uniqueness, $g(z) \equiv 0$.

So we have $\forall \delta > 0 \ \exists N \ \forall k \geq N \ |f(iy+n_k)| < \delta$ (uniformly in y).

$n_k \to \infty$, it follows $f(x+iy) \to 0$ as $x \to \infty$ uniformly in y, $y \in [-1+\varepsilon, 1-\varepsilon]$.

#7 $\mathcal{D}$ is bounded, $f(z): \mathcal{D} \to \mathcal{D}$ analytic, z_0 is fixed point of f.

Consider $f_n(z) = f^{\circ n}(z)$, analytic on $\mathcal{D}$, uniformly bounded on $\mathcal{D}$. By Montel's, $\exists n_k$ s.t. $f_{n_k}(z) \to g(z)$ uniformly on compact subsets; pick $\{|z-z_0| \leq \varepsilon\} \subseteq \mathcal{D}$ compact in $\mathcal{D}$

Then $f_{n_k}^{(m)}(z) \to g^{(m)}(z)$ uniformly on $\{|z-z_0| \leq \varepsilon/2\}$, in particular, $f_{n_k}^{(m)}(z_0) \to g^{(m)}(z_0)$. (Actually, we need $m=1$).

$f_n'(z) = f'(f_{n-1}(z)) \cdot f'(f_{n-2}(z)) \cdots f'(z)$ by chain rule

$f_n'(z_0) = f'(f_{n-1}(z_0)) \cdot f'(f_{n-2}(z_0)) \cdots f'(z_0) = [f'(z_0)]^n$

if $|f'(z_0)| > 1$, then $|f'(z_0)|^{n_k} \to \infty$ and $f_{n_k}'(z_0)$ cannot converge. Contradiction.

XI.6 #1 $g(z) = \dfrac{z-b}{1-\bar{b}z}$ $\quad f(z) = \dfrac{z - h(g(0))}{1-\overline{h(g(0))}z}$

$h(z)$ is a branch of $\sqrt{z}$.

$\psi(z) := f(h(g(z)))$; $\psi'(z) = f'(h(g(z)))\cdot h'(g(z))\cdot g'(z)$

$\psi'(0) = f'(h(g(0)))\cdot h'(g(0))\cdot g'(0)$.

Note that $g'(z) = \dfrac{1-|b|^2}{(1-\bar{b}z)^2}$, $f'(z) = \dfrac{1-|h(g(0))|^2}{(1-\overline{h(g(0))}z)^2}$, so

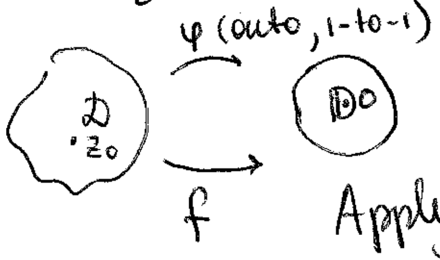
$$\psi'(0) = \frac{1-|h(g(0))|^2}{(1-|h(g(0))|^2)^2}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{-b}}(1-|b|^2) =$$

$$= \frac{1}{1-|b|}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{-b}}(1-|b|^2) = \frac{1}{2}\,\frac{1+|b|}{\sqrt{-b}}$$

$|\psi'(0)| = \dfrac{1}{2}\,\dfrac{1+|b|}{\sqrt{|b|}} = \dfrac{1}{2}\left(t+\dfrac{1}{t}\right)$, where $t = \sqrt{|b|} < 1$

$t + \frac{1}{t} > 2\sqrt{t\cdot\frac{1}{t}} = 2$, because $t \neq \frac{1}{t}$.

#2



Consider $f(\varphi^{-1}(z))$, maps $\mathbb{D}$ in $\mathbb{D}$

Apply Pick's lemma to $f\circ\varphi^{-1}$ at 0:

$$\left|(f\circ\varphi^{-1})'(0)\right| \le \frac{1-|f(\varphi^{-1}(0))|^2}{1-|0|^2} \qquad |f'(z_0)|\cdot\frac{1}{\varphi'(z_0)} \le 1-|f(z_0)| \le 1$$

So $|f'(z_0)| \le \varphi'(z_0)$

Equality holds iff 1) $|f(z_0)| = 0$ and 2) $f\circ\varphi^{-1}$ is conformal self-map of $\mathbb{D}$. 1) $\Rightarrow$ $f\circ\varphi^{-1}$ fixed 0, so with 2) we get

$(f\circ\varphi^{-1})(z) = e^{i\theta}\cdot z \Leftrightarrow f(z) = e^{i\theta}\cdot\varphi(z)$.

#3 By #2 $|f'(z_0)| \leq \varphi'(z_0)$, so $\mathrm{Re}\, f'(z_0) \leq |f'(z_0)| \leq \varphi'(z_0)$.

The equality holds iff both equalities hold above,
i.e. when $f'(z_0)$ is real and positive and
$f(z) = e^{i\theta} \cdot \varphi(z_0)$

Then $f'(z_0) = e^{i\theta} \cdot \varphi'(z_0)$, both sides positive real,
so $\theta = 2\pi n$, $f(z) = \varphi(z)$.